

Differential groupoids and C^* -algebras.

Piotr Stachura

Departement of Mathematical Methods in Physics,
Faculty of Physics, University of Warsaw,
ul. Hoza 74, 00-682 Warszawa
Poland
e-mail: stachura@fuw.edu.pl

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Abstract

For a differential groupoid Γ we construct a C^* algebra $C^*(\Gamma)$ in a way that the correspondance $\Gamma \longrightarrow C^*(\Gamma)$ is a covariant functor from the category of differential groupoids in a sense of S. Zakrzewski to the category of C^* algebras.

1 Introduction

In this paper we construct a covariant functor from the category of differential groupoids to the category of C^* algebras in the sense of [8]. However our definition of morphism of differential groupoids is different from the standard one i.e. a mapping which satisfies obvious compatibility condition with respect to groupoid structure. Let us argue that there is rather no hope to construct such a functor with a standard notion of morphism. The main problem can be shown in the discrete case so let us assume that all sets have discrete topology.

Let (Γ, m, E, s) be a groupoid (see the next section for the notation) and let $\mathcal{A}(\Gamma)$ denote linear space of complex functions with compact support. (i.e. for $f \in \mathcal{A}(\Gamma)$ we have $f(x) \neq 0$ only for finite number of x 's). There is a natural notion of convolution and star operation in $\mathcal{A}(\Gamma)$ which make it a $*$ -algebra. Namely

$$(f_1 f_2)(x) := \sum_{yz=x} f_1(y)f_2(z) = \sum_{y \in F_l(x)} f_1(y)f_2(s(y)x) = \sum_{z \in F_r(x)} f_1(xs(z))f_2(z), \quad f^*(x) := \overline{f(s(x))}$$

$(F_l(x), F_r(x))$ denote left and right fiber containing x . We expect that $C^*(\Gamma)$ will be a completion of $\mathcal{A}(\Gamma)$ with respect to some C^* -norm.

The “extremal” examples of groupoids are sets and groups. For sets the above multiplication is equal to a pointwise multiplication and for groups it is usual convolution. The standard definition of morphism of groupoids reduces to a mapping if groupoids are sets and to a group homomorphism if they are groups. If $h : \Gamma \longrightarrow \Gamma'$ is a group homomorphism, we can *push forward* convolution algebra by a formula $(\hat{h}f)(x') := \sum_{x \in h^{-1}(x')} f(x)$, which defines mapping $\hat{h} : \mathcal{A}(\Gamma) \longrightarrow \mathcal{A}(\Gamma')$. But if $h : \Gamma \longrightarrow \Gamma'$ is a mapping of sets, functions with pointwise multiplication can be *pulled back* by $(\hat{h}f')(x) := f'(h(x))$. In fact $\hat{h}f'$ can have non compact support, but this is not a problem, since we know that it should belong to a (some kind of) multiplier algebra of $\mathcal{A}(\Gamma)$ so $(\hat{h}f')f$ should be in $\mathcal{A}(\Gamma)$ for any $f \in \mathcal{A}(\Gamma)$ and certainly this is true. Disregarding the subtlety in this case have $\hat{h} : \mathcal{A}(\Gamma') \longrightarrow \mathcal{A}(\Gamma)$. We expect that $C^*(h)$ will be some extension of \hat{h} . And here we are in trouble, since our “ C^* -functor” is covariant in the first case and contravariant in the second. So to achieve our goal we *need a definition of morphism between groupoids which reduces to a group homomorphism if groupoids are groups and to a mapping in the reverse direction if groupoids are sets*. In particular this suggest that morphisms should be rather *relations* instead of *mappings*. Such a definition was given in [1] and extended to a differential setting in [2]. Let us briefly explain the main idea of the construction, still in the discrete setting. We suggest to look at the next section before the following.

Let (Γ, m, E, s) and (Γ', m', E', s') be groupoids. A morphism from Γ to Γ' is a relation, which satisfies some obvious compatibility conditions. In particular it turns out that it defines a mapping $f_h : E' \longrightarrow E$

and for each $b \in E'$ a mapping $h_b^R : F_r(f_h(b)) \longrightarrow F_r(b)$. For a morphism $h : \Gamma \longrightarrow \Gamma'$ and $f \in \mathcal{A}(\Gamma)$ we define $\hat{h}f$ – a linear mapping on $\mathcal{A}(\Gamma')$ by the formula:

$$((\hat{h}f)f')(x') := \sum_{x \in F_l(b)} f(x)f'(s'(h_a^R(x))x'),$$

where $a := e'_L(z)$, $b := f_h(a)$. By the same formula we define $\pi_h(f)f'$ where we view f' as an element of $L^2(\Gamma')$ – the Hilbert space of square summable functions on Γ' . Let us also define norms on $\mathcal{A}(\Gamma)$: $\|f\|_l := \sup_{a \in E} \sum_{x \in F_l(a)} |f(x)|$, $\|f\|_r := \sup_{a \in E} \sum_{x \in F_r(a)} |f(x)|$, and $\|f\| := \max\{\|f\|_l, \|f\|_r\}$. It is not difficult to prove the following:

Proposition 1.1 *a) $(\mathcal{A}(\Gamma), *, \|\cdot\|)$ is a normed *-algebra.*

*b) $\|\pi_h(f)\| \leq \|f\|$ and π_h is a representation of a *-algebra $\mathcal{A}(\Gamma)$*

c) $f_3^(\hat{h}(f_1)f_2) = (\hat{h}(f_1^*)f_3)^*f_2$ for any $f_1 \in \mathcal{A}(\Gamma)$, $f_2, f_3 \in \mathcal{A}(\Gamma')$.*

d) If $k : \Gamma' \longrightarrow \Gamma''$ is a morphism of groupoids then:

$\pi_k(\hat{h}(f_1)f_2)f_3 = \pi_{kh}(f_1)\pi_k(f_2)f_3$ for any $f_1 \in \mathcal{A}(\Gamma)$, $f_2 \in \mathcal{A}(\Gamma')$, $f_3 \in L^2(\Gamma'')$ – with compact support.

$\hat{k}(\hat{h}(f_1)f_2)f_3 = \hat{k}h(f_1)(\hat{k}(f_2)f_3)$ for any $f_1 \in \mathcal{A}(\Gamma)$, $f_2 \in \mathcal{A}(\Gamma')$, $f_3 \in \mathcal{A}(\Gamma'')$.

Using these facts one can define the C^* norm on $\mathcal{A}(\Gamma)$ by: $\|f\|_{C^*} := \sup \|\pi_h(f)\|$ where the supremum is taken over all morphism $h : \Gamma \longrightarrow \Gamma'$. (This is obviously C^* -seminorm, but one can show that there exists faithful representation of $\mathcal{A}(\Gamma)$.) The completion of $\mathcal{A}(\Gamma)$ with respect to this norm is C^* – algebra of Γ and one can see that \hat{h} extends to a $C^*(h) \in \text{Mor}(C^*(\Gamma), C^*(\Gamma'))$. The extension of this construction to a differential setting is the main result of the paper.

Of course in the above case we can also proceed in the standard way: first one can complete $(\mathcal{A}(\Gamma), *, \|\cdot\|)$ to get a Banach *-algebra and then take envelopping C^* algebra. However, in such construction the functoriality is lost and moreover it seems that there is no natural, geometric norm on $\mathcal{A}(\Gamma)$ in the differential case, so we don't have Banach algebra.

Let us now say a few words about our motivations. One is to get “geometric models” of quantum groups, especially non compact, from double Lie groups. For a given double Lie group $(G; A, B)$ (see section 3) one can define diffeomorphism $\Psi : G \times G \ni (x, y) \mapsto (x(a_L(y))^{-1}, b_R(x(a_L(y))^{-1})y) \in G \times G$, which satisfies pentagonal equation: $\Psi_{23}\Psi_{12} = \Psi_{12}\Psi_{13}\Psi_{23}$, where $\Psi_{23} : G \times G \times G \ni (x, y, z) \mapsto (x, \Psi(y, z)) \in G \times G \times G$, etc. Since Ψ is a diffeomorphism, it defines (by a push-forward) unitary, multiplicative operator [11] on $L^2(G \times G) = L^2(G) \otimes L^2(G)$ ($L^2(G)$ is a Hilbert space of square integrable half densities on G). This operator is manageable in the sense of Woronowicz [7],[10] so it defines a quantum group.

From the other side if $(G; A, B)$ is a double Lie group we can define two differential groupoids structures on G : $G_A := (G, m_A, A, s_A)$ and $G_B := (G, m_B, B, s_B)$. It turns out that m_B^T is a morphism $G_A \longrightarrow G_A \times G_A$ which is coassociative: $(m_B^T \times id)m_B^T = (id \times m_B^T)m_B^T$. Applying our C^* functor we get a coassociative morphism $\Delta \in \text{Mor}(C^*(G_A), C^*(G_A \times G_A))$. We expect that $C^*(G_A \times G_A)$ be (a some sort of) $C^*(G_A) \otimes C^*(G_A)$. In this way we get one of the main ingredients of quantum group structure on $C^*(G_A)$. It seems that also other ingredients as defined in [6] have natural geometric interpretation in the groupoid setting. The details will be presented elsewhere.

For connections with symplectic geometry and quantisation see Appendix A.

Sections 2 and 3 are of introductory character and are more or less contained in [1],[2]. We hope that they make the paper as self contained as possible, keeping its size limited. Section 2 is devoted to algebraic structure of groupoid in the language of relations. In section 3 we add differential structure. In section 4 – the central of the paper, we construct for any morphism of differential groupoids h the mappings \hat{h}, π_h with the properties given above. In section 5 we define a C^* algebra of a differential groupoid and prove the functorial properties of the construction. We also interpret bisections and functions on the set of identities as multipliers. In Appendices B and C we relate cocycles on Γ with one parameter groups and give a construction of weights on $C_{red}^*(\Gamma)$. In Appendix D we discuss some basic facts about subgroupoids and Appendix E contains necessary results from theory of C^* -algebras.

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2 Groupoids – algebraic structure

We begin by recalling some facts about relations.

A *relation* r from X to Y is a triple $r = (R; Y, X)$, where X and Y are sets and R is a subset of $Y \times X$. R is a *graph* of r and we denote it by $Gr(r)$. A relation r from X to Y will be denoted by $r : X \multimap Y$ (note the special type of arrow). Relations can be composed, if $s : X \multimap Y$ and $r : Y \multimap Z$, then a composition rs is a relation from X to Z defined by $Gr(rs) := \{(z, x) \in Z \times X : \exists y \in Y [(z, y) \in Gr(r) \text{ and } (y, x) \in Gr(s)]\}$. We say that the composition rs is *simple* iff for any $(z, x) \in Gr(rs)$ there exists unique $y \in Y$ such that $(y, x) \in Gr(s)$ and $(z, y) \in Gr(r)$.

If $r : X \multimap Y$ then its *transposition* is a relation $r^T : Y \multimap X$ with $Gr(r^T) := \{(x, y) \in X \times Y : (y, x) \in Gr(r)\}$. The cartesian product of relation is also naturally defined: if $r : X \multimap Y$ and $s : Z \multimap T$ then $r \times s : X \times Z \multimap Y \times T$ is a relation with graph $Gr(r \times s) := \{(y, t, x, z) \in Y \times T \times X \times Z : (y, x) \in Gr(r) \text{ and } (t, z) \in Gr(s)\}$.

If $r : X \multimap Y$ and $A \subset X$ we denote by $r(A)$ the image of A by r : $r(A) := \{y \in Y : \exists x \in A (y, x) \in Gr(r)\}$. Let $\{1\}$ denote one point set. Now we can formulate the basic definition:

Definition 2.1 [1] *Groupoid* is a quadruple (Γ, m, e, s) where Γ is a set, $m : \Gamma \times \Gamma \multimap \Gamma$ and $e : \{1\} \multimap \Gamma$ are relations, $s : \Gamma \multimap \Gamma$ is an involution which satisfy:

- associativity* $m(m \times id) = m(id \times m)$,
- identity* $m(e \times id) = m(id \times e) = id$,
- inverse* $sm = m(s \times s) \sim$ where $\sim : \Gamma \times \Gamma \ni (x, y) \mapsto (y, x) \in \Gamma \times \Gamma$,
- strong positivity* for any $x \in \Gamma$ $\emptyset \neq m(s(x), x) \subset e(\{1\})$

Relation m is called *multiplication*, s - *inverse* and $E := e(\{1\})$ - the set of *identities*.

Remark 2.2 Notice that first three conditions are formally the same as in a group case but instead of mappings we use relations. The above definition is equivalent (cf. proposition below) to the "ordinary" definition of groupoid: *A groupoid is a small category in which every morphism is an isomorphism*. But if we think of groupoid as of category, the natural candidates for morphism are functors—this is not our point of view, so we prefer the definition based on relations.

Proposition 2.3 [1] *Let (Γ, m, e, s) be a groupoid. Then:*

- a) *If $a, b \in E$ then $m(a, b) \neq \emptyset$ iff $a = b$ and in this case $m(a, a) = a$.*
- b) *There exist unique mappings $e_L, e_R : \Gamma \longrightarrow E$ such that $m(e_L(x), x) = x = m(x, e_R(x))$ for any $x \in \Gamma$ and $e_L(a) = e_R(a) = a$ for any $a \in E$.*
- c) *$m(s(x), x) = e_R(x)$, $m(x, s(x)) = e_L(x)$*
- d) *$m(x, y) \neq \emptyset$ iff $e_R(x) = e_L(y)$*
- e) *$m(x, y) \cap E \neq \emptyset$ implies $y = s(x)$.*
- f) *$m(x, y)$ consists of at most one point.*

Proof: We prove these statements here as an exercise in dealing with relations.

a) From the *identity* axiom: if $(x; a, y) \in m$ for some $a \in E$ then $x = y$ and if $(x; y, b) \in m$ for some $b \in E$ then $x = y$. So for $a, b \in E$, if $(x; a, b) \in m$ then $x = a = b$. Also for any $x \in \Gamma$ there exists some $b \in E$ such that $(x; b, x) \in m$, so for any $b \in E$ $(b; b, b) \in m$. Moreover from the *strong positivity* we have $s(a) = a$ for $a \in E$.

b) Suppose that $(x; b_1, x) \in m$ and $(x; b_2, x) \in m$ for $b_1, b_2 \in E$. Then $(x; b_1, b_2, x) \in m(id \times m) = m(m \times id)$ (*associativity*). This means that $m(b_1, b_2)$ is not empty and then $b_1 = b_2$. This proves the existence and uniqueness of e_L . In the same way one deals with e_R .

c) From *strong positivity* we have: $(a; s(x), x) \in m$ for $a \in E$ so $(a; s(x), x, e_R(x)) \in m(id \times m) = m(m \times id)$. From this it follows that exist (x_1, x_2) such that $(a; x_1, x_2) \in m$ and $(x_1, x_2; s(x), x, e_R(x)) \in (m \times id)$. Then $x_2 = e_R(x)$ and $a = x_1 = e_R(x)$. So $m(s(x), x)$ is one point: $e_R(x)$. If $(e_R(x); s(x), x) \in m$ then $(e_R(x); s(x), e_L(x), x) \in m(id \times m) = m(m \times id)$ so $(e_R(x); x_1, x_2) \in m$ and $(x_1, x_2; s(x), e_L(x), x) \in (m \times id)$. From this we infer that $x_2 = x$ and $(x_1; s(x), e_L(x)) \in m$ so $e_R(s(x)) = e_L(x)$ and $e_L(x) = xs(x)$ since s is an involution.

d) If $(z; x, y) \in m$ then $(z; x, e_R(x), y) \in m(m \times id) = m(id \times m)$ so $m(e_R(x), y)$ is not empty and $e_R(x) =$

$e_L(y)$. Conversely, if $e_R(x) = e_L(y)$ then $(x; x, e_R(x) = e_L(y)) \in m$, so $(x; x, y, s(y)) \in m(id \times m) = m(m \times id)$ and $m(x, y)$ is not empty.

e) If $(a; x, y) \in m$ for some $a \in E$ then $(a; e_L(x), x, y) \in m(m \times id) = m(id \times m)$ so $(a; e_L(x), x_2) \in m$ and $(x_2; x, y) \in m$ so $a = e_L(x)$. In the same way $a = e_R(y) = e_L(s(y))$. From this fact: $(s(y); a, s(y)) \in m$ and $(s(y); x, y, s(y)) \in m(m \times id) = m(id \times m)$. We have: $(s(y); x_1, x_2) \in m$ and $(x_1, x_2; x, y, s(y)) \in (id \times m)$ so $(s(y); x, e_L(y) = e_R(x)) \in m$ so $s(y) = x$.

f) Let $(x_1; x, y) \in m$ and $(x_2; x, y) \in m$. In this situation $e_L(x_1) = e_R(s(x_2))$ so $(z; x_1, s(x_2)) \in m$ for some z . So $(z; x, y, s(x_2)) \in m(m \times id)$ and $(z; x, x_4) \in m$ and $(x_4; y, s(x_2)) \in m$. But then $(x_4; y, s(y), s(x)) \in m(id \times m)$ and $x_4 = s(x)$. So $z \in E$ and $x_1 = x_2$.

■

Now we explain our notation.

The set of composable pairs will be denoted by $\Gamma^{(2)} := m^T(\Gamma) = \{(x, y) \in \Gamma \times \Gamma : e_R(x) = e_L(y)\}$. From the statements c) and f) of the above proposition follows that m restricted to $\Gamma^{(2)}$ is a surjective mapping on Γ . The set of identities E will also be denoted by Γ^0 . If it doesn't lead to any confusion we write $x = x_1 x_2$ instead of $(x; x_1, x_2) \in Gr(m)$. For $A, B \subset \Gamma$ let $AB := \{ab : a \in A, b \in B\}$. If $r : X \twoheadrightarrow Y$ we also write $(y, x) \in r$ instead of $(y, x) \in Gr(r)$.

For $x \in \Gamma$, by $F_l(x)$ and $F_r(x)$ we denote left and right fibers containing x i.e. $F_l(x) := e_L^{-1}(e_L(x))$ and $F_r(x) := e_R^{-1}(e_R(x))$. If $a \in E$ we also write ${}_a\Gamma := F_l(a)$ and $\Gamma_a := F_r(a)$. Clearly ${}_a\Gamma \cap \Gamma_a$ is a group.

For $x \in \Gamma$ let $L_x : F_l(e_R(x)) \ni z \mapsto xz \in F_l(x)$ and $R_x : F_r(e_L(x)) \ni z \mapsto zx \in F_r(x)$ denote left and right translation by x . They are bijections.

Examples 2.4 a) *Sets*. If X is a set then (X, d^T, X, id) , where $d : X \longrightarrow X \times X$ is a diagonal mapping, is a groupoid. Conversely, any groupoid such that m^T is a mapping is of this type.

b) *Groups*. If G is a group then $(G, m_G, \{e\}, s)$, where m_G, s are group multiplication and group inverse, is a groupoid and any groupoid for which m is a mapping is a group.

c) *Pair groupoids*. Examples a) and b) are “extremal” examples of groupoids. The “middle” and the simplest are pair groupoids. Let X be a set. We put: $\Gamma := X \times X$, $Gr(m) := \{((x, y); (x, z), (z, y)) : x, y, z \in X\}$, $s(x, y) := (y, x)$ and $\Gamma^0 := \{(x, x) : x \in X\}$. Then (Γ, m, Γ^0, s) is a groupoid.

d) *Equivalence relations*. If $R \subset X \times X$ is an equivalence relation on X , then (R, m, Γ^0, s) where m, Γ^0, s are as above is a groupoid.

e) *Transformation groupoids*. Let a group G acts on a set X . We denote the action by $G \times X \ni (g, x) \mapsto gx \in X$. Define $\Gamma := G \times X$, $s(g, x) := (g^{-1}, gx)$, $E := \{e\} \times X$ and m by

$$Gr(m) := \{((g_1 g_2, x); (g_1, g_2 x), (g_2, x)) : g_1, g_2 \in G, x \in X\} \subset \Gamma \times \Gamma \times \Gamma.$$

Then (Γ, m, E, s) is a groupoid.

f) *Double groups*. [1] Let $(G; A, B)$ be double group i.e. $A, B \subset G$ are subgroups, $A \cap B = \{e\}$ and $G = AB$. In this situation each element of G can be written uniquely as: $g = a_L(g)b_R(g) = b_L(g)a_R(g)$. This decomposition defines four mappings: $a_R, a_L : G \longrightarrow A$ and $b_R, b_L : G \longrightarrow B$. Let $m_A : G \times G \twoheadrightarrow G$ be a relation defined by

$$m_A(g_1, g_2) := \begin{cases} g_1 b_R(g_2) = b_L(g_1) g_2 & \text{if } a_R(g_1) = a_L(g_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Let $s_A : G \ni g \mapsto (b_L(g))^{-1} a_L(g) \in G$. Then $G_A := (G, m_A, A, s_A)$ is a groupoid. The same holds for $G_B := (G, m_B, B, s_B)$.

g) And many, many more. See e.g. [5]

The cartesian product of groupoids is defined in a natural way. For groupoids $(\Gamma_i, m_i, E_i, s_i)$, $i = 1, 2$, their cartesian product, which we denote simply by $\Gamma_1 \times \Gamma_2$, is a groupoid

$$(\Gamma_1 \times \Gamma_2, (m_1 \times m_2)(id \times \sim \times id), E_1 \times E_2, s_1 \times s_2).$$

Morphisms of groupoids.

Definition 2.5 [1] Let (Γ, m, e, s) and (Γ', m', e', s') be groupoids. A *morphism* from Γ to Γ' is a relation $h : \Gamma \multimap \Gamma'$ such that:

1. $hm = m'(h \times h)$
2. $hs = s'h$
3. $he = e'$.

Note the following proposition [1]:

Proposition 2.6 Let $h : \Gamma \multimap \Gamma'$ be a morphism of groupoids. Then:

a) The compositions in the definition above are simple.

b) Let the relation $h_0 : E \multimap E'$ be defined by $Gr(h_0) := Gr(h) \cap (E' \times E)$.

Then $(e'_R \times e_R)Gr(h) = (e'_L \times e_L)Gr(h) = Gr(h_0)$ and $f_h := h_0^T$ is a mapping.

c) Let $b \in E'$ and $a := f_h(b)$. Let us define two relations $h_b^R : F_r(a) \multimap F_r(b)$ and $h_b^L : F_l(a) \multimap F_l(b)$ by $Gr(h_b^R) := Gr(h) \cap (F_r(b) \times F_r(a))$ and $Gr(h_b^L) := Gr(h) \cap (F_l(b) \times F_l(a))$. Then h_b^R, h_b^L are mappings. ■

Morphisms can also be characterised in terms of mapping. This is the contents of the following:

Proposition 2.7 Any morphism $h : \Gamma \multimap \Gamma'$ determines and is uniquely determined by mappings $f : E' \rightarrow E$ and $g : \Gamma \times_f E' \rightarrow \Gamma'$, where $\Gamma \times_f E' := \{(x, e') \in \Gamma \times E' : e_R(x) = f(e')\}$ which satisfy conditions:

a) $e_L e_R^{-1}(f(E')) = f(E')$ (then also $e_R e_L^{-1}(f(E')) = f(E')$)

b) $e'_R g(x, e') = e'$

c) $s'g(x, e') = g(s(x), e'_L g(x, e'))$

d) $\forall x_1, x \in \Gamma [(x, e') \in \Gamma \times_f E' \text{ and } e_R(x_1) = e_L(x)] \Rightarrow g(x_1 x, e') = g(x_1, e'_L g(x, e')) g(x, e')$

Proof: First, we show that such two mappings define morphism of groupoids. Let a relation h be given by the graph: $Gr(h) := \{(g(x, e'), x) : (x, e') \in \Gamma \times_f E'\}$. Notice that

$$e' = e'_R g(f(e'), e') = s'(g(f(e'), e')) g(f(e'), e') = g(f(e'), e'_L g(f(e'), e')) g(f(e'), e') = g(f(e'), e').$$

First equality follows from statement b), third from c) and the last one from d). This shows that relation h satisfies: $hE = E'$.

We have the following sequence of equivalences:

$$\begin{aligned} (y, x) \in hs &\iff (y, s(x)) \in h \iff y = g(s(x), e') \iff s'(y) = g(x, e'_L g(s(x), e')) \iff \\ &(s'(y), x) \in h \iff (y, x) \in s'h. \end{aligned}$$

In this way $hs = s'h$.

Let $(y; x_1, x_2) \in hm$. Now we have:

$$(y; x_1, x_2) \in hm \iff [e_R(x_1) = e_L(x_2) \text{ and } (y, x_1 x_2) \in h] \iff [e_R(x_1) = e_L(x_2) \text{ and } y = g(x_1 x_2, e')].$$

It follows that $e_R(x_2) = f(e')$, so for $y_2 := g(x_2, e')$ and $y_1 := g(x_1, e'_L(y_2))$ we have: $(y_2, x_2) \in h$, $(y_1, x_1) \in h$ and $y = y_1 y_2$. From this: $(y; x_1, x_2) \in m'(h \times h)$.

Conversely, for $(y; x_1, x_2) \in m'(h \times h)$ we have $y = y_1 y_2$ with $e'_R(y_1) = e'_L(y_2)$ and $y_1 = g(x_1, e'_1)$, $y_2 = g(x_2, e'_2)$. Now $s'(y_2) = g(s(x_2), e'_L(y_2)) = g(s(x_2), e'_R(y_1))$, so $e_L(x_2) = e_R(x_1)$ and x_1, x_2 are composable. So we get that $m'(h \times h) \subset hm$.

Now the “determines” part. If $h : \Gamma \multimap \Gamma'$ is a morphism, define $f := f_h$ and $g(x, e') := h_{e'}^R(x)$. Then a) and b) follows directly from prop. 2.6. Let us show c).

$$\begin{aligned} y = s'g(x, e') &\iff s'(y) = g(x, e') \iff [(s'(y), x) \in h \text{ and } e'_R s'(y) = e'_L(y) = e'] \iff \\ &[(y, s(x)) \in h \text{ and } e'_L(y) = e']. \end{aligned}$$

But $e'_R(y) = e'_L g(x, e')$, so $y = g(s(x), e'_L g(x, e'))$.

d) Let $e_R(x) = f(e')$, $e_R(x_1) = e_L(x)$, $y := g(x, e')$ and $z := g(x_1 x, e')$. This means that y is given by the conditions: $(y, x) \in h$, $e'_R(y) = e'$ and z by $(z, x_1 x) \in h$, $e'_R(z) = e'$. So $z, s'(y)$ are composable and $(zs'(y), x_1 x, s(x)) \in m'(h \times h) = hm$. From this we have $(zs'(y), x_1) \in h$, $e'_R(zs'(y)) = e'_L(y)$, $zs'(y) = g(x_1, e'_L g(x, e'))$ and $z = g(x_1, e'_L g(x, e')) y$. ■

Let $h : \Gamma \longrightarrow \Gamma'$ be a morphism and f, g be as above. Denote $\tilde{\Gamma} := \Gamma \times_f E'$ and define

$$\tilde{s} : \tilde{\Gamma} \ni (x, b) \mapsto (s(x), e'_L g(x, b)) \in \tilde{\Gamma}, \quad \tilde{E} := E \times_f E'$$

and a relation $\tilde{m} : \tilde{\Gamma} \times \tilde{\Gamma} \longrightarrow \tilde{\Gamma}$ by:

$$Gr(\tilde{m}) := \{(x_1 x_2, b_2; x_1, e'_L g(x_2, b_2), x_2, b_2) : e_R(x_1) = e_L(x_2), (x_2, b_2) \in \tilde{\Gamma}\}$$

Lemma 2.8 $(\tilde{\Gamma}, \tilde{m}, \tilde{E}, \tilde{s})$ is a groupoid.

Proof: Let us first check that \tilde{s} is an involution:

$$\tilde{s}\tilde{s}(x, b) = \tilde{s}(s(x), e'_L g(x, b)) = (x, e'_L g(s(x), e'_L g(x, b))) = (x, e'_R s'g(s(x), e'_L g(x, b))) = (x, e'_R g(x, b)) = (x, b).$$

1. $\tilde{m}(\tilde{m} \times id) = \tilde{m}(id \times \tilde{m})$.

Compute the left hand side:

$$\begin{aligned} (x_1, b_1; x_2, b_2, x_3, b_3, x_4) \in \tilde{m}(\tilde{m} \times id) &\iff \\ \exists (x_5, b_5), (x_6, b_6) : [(x_1, b_1; x_5, b_5, x_6, b_6) \in \tilde{m} \text{ and } (x_5, b_5, x_6, b_6; x_2, b_2, x_3, b_3, x_4, b_4) \in (\tilde{m} \times id)] &\iff \\ \exists (x_5, b_5) : [(x_1, b_1; x_5, b_5, x_4, b_4) \in \tilde{m} \text{ and } (x_5, b_5; x_2, b_2, x_3, b_3) \in \tilde{m}] &\iff \\ [e_R(x_2) = e_L(x_3) \text{ and } b_2 = e'_L g(x_3, b_3) \text{ and } (x_1, b_1; x_2 x_3, b_3, x_4, b_4) \in \tilde{m}]. \end{aligned}$$

So $Gr(\tilde{m}(\tilde{m} \times id)) =$

$$= \{(x_2 x_3 x_4, b_4; x_2, e'_L g(x_3, e'_L g(x_4, b_4)), x_3, e'_L g(x_4, b_4), x_4, b_4) : e_R(x_2) = e_L(x_3), e_R(x_3) = e_L(x_4), (x_4, b_4) \in \tilde{\Gamma}\}.$$

And the right hand side:

$$\begin{aligned} (x_1, b_1; x_2, b_2, x_3, b_3, x_4) \in \tilde{m}(id \times \tilde{m}) &\iff \\ \exists (x_5, b_5), (x_6, b_6) : [(x_1, b_1; x_5, b_5, x_6, b_6) \in \tilde{m} \text{ and } (x_5, b_5, x_6, b_6; x_2, b_2, x_3, b_3, x_4, b_4) \in (id \times \tilde{m})] &\iff \\ [e_R(x_3) = e_L(x_4) \text{ and } b_3 = e'_L g(x_4, b_4) \text{ and } (x_1, b_1; x_2, b_2, x_3 x_4, b_4) \in \tilde{m}]. \end{aligned}$$

It follows that $Gr(\tilde{m}(id \times \tilde{m})) =$

$$= \{(x_2 x_3 x_4, b_4; x_2, e'_L g(x_3 x_4, b_4), x_3, e'_L g(x_4, b_4), x_4, b_4) : e_R(x_2) = e_L(x_3), e_R(x_3) = e_L(x_4), (x_4, b_4) \in \tilde{\Gamma}\}.$$

But $e'_L g(x_3 x_4, b_4) = e'_L (g(x_3, e'_L g(x_4, b_4)) g(x_4, b_4)) = e'_L g(x_3, e'_L g(x_4, b_4))$. In this way $\tilde{m}(\tilde{m} \times id) = \tilde{m}(id \times \tilde{m})$.

2. $\tilde{m}(\tilde{E} \times id) = \tilde{m}(id \times \tilde{E}) = id$.

If $(x, b; a_1, b_1, x_2, b_2) \in \tilde{m}$ for some $(a_1, b_1) \in \tilde{E}$ then $x = a_1 x_2$ and $b = b_2$ so $(x, b) = (x_2, b_2)$. Conversely for any $(x, b) \in \tilde{\Gamma}$ we have $(x, b; e_L(x), e'_L g(x, b), x, b) \in \tilde{m}$. So $\tilde{m}(\tilde{E} \times id) = id$. In the same way one shows that $\tilde{m}(id \times \tilde{E}) = id$.

3. $\tilde{s}\tilde{m} = \tilde{m}(\tilde{s} \times \tilde{s}) \sim$.

The left hand side:

$$\begin{aligned} (x, b; x_1, b_1, x_2, b_2) \in \tilde{s}\tilde{m} &\iff (s(x), e'_L g(x, b); x_1, b_1, x_2, b_2) \in \tilde{m} \iff \\ e_R(x_1) = e_L(x_2), x = s(x_2)s(x_1), e'_L g(x, b) = b_2, b_1 = e'_L g(x_2, b_2) \end{aligned}$$

but $b = e'_R g(x, b) = e'_L g(s(x), e'_L g(x, b)) = e'_L g(x_1 x_2, b_2) = e'_L g(x_1, e'_L g(x_2, b_2))$.

In this way

$$Gr(\tilde{s}\tilde{m}) = \{(s(x_2)s(x_1), e'_L g(x_1, e'_L g(x_2, b_2)); x_1, e'_L g(x_2, b_2), x_2, b_2) : e_R(x_1) = e_L(x_2), (x_2, b_2) \in \tilde{\Gamma}\}.$$

And the right hand side:

$$\begin{aligned} (x, b; x_1, b_1, x_2, b_2) \in \tilde{m}(\tilde{s} \times \tilde{s}) &\sim \iff (x, b; s(x_2), e'_L g(x_2, b_2), s(x_1), e'_L g(x_1, b_1)) \in \tilde{m} \iff \\ x = s(x_2)s(x_1), b = e'_L g(x_1, b_1), e_R(x_1) = e_L(x_2), e'_L g(x_2, b_2) = e'_L g(s(x_1), e'_L g(x_1, b_1)) = e'_L s'g(x_1, b_1) = b_1. \end{aligned}$$

So we have: $Gr(\tilde{s}\tilde{m}) = Gr(\tilde{m}(\tilde{s} \times \tilde{s}) \sim)$.

4. $(s(x)x, b; s(x), e'_L g(x, b), x, b) \in \tilde{m}$ for any $(x, b) \in \tilde{\Gamma}$ and $(s(x)x, b) \in \tilde{E}$.

■

Consider relations $h_1 : \Gamma \longrightarrow \tilde{\Gamma}$ and $h_2 : \tilde{\Gamma} \longrightarrow \Gamma'$ defined by: $Gr(h_1) := \{(x, b; x) : (x, b) \in \tilde{\Gamma}\}$, $Gr(h_2) := \{(g(x, b); x, b) : (x, b) \in \tilde{\Gamma}\}$. Clearly we have $h = h_2 h_1$, moreover h_1 is a morphism from Γ to $\tilde{\Gamma}$ and h_2 is a morphism from $\tilde{\Gamma}$ to Γ' . For h_1 the mappings between fibers are bijective, and f_{h_2} is bijective mapping. In this way we have the following structure:

Proposition 2.9 *If $h : \Gamma \longrightarrow \Gamma'$ is a morphism of groupoids, then exists groupoid $\tilde{\Gamma}$, morphisms $k : \Gamma \longrightarrow \tilde{\Gamma}$ and $l : \tilde{\Gamma} \longrightarrow \Gamma'$ such that $h = lk$ and:*

- a) *For each $a \in \tilde{E}$ the mappings k_a^R and k_a^R are bijections.*
- b) *l is a mapping from $\tilde{\Gamma} \longrightarrow \Gamma'$ which is bijective when restricted to \tilde{E} .*

■

Groupoids together with just defined morphisms form a category as the following lemma states.

Lemma 2.10 [1] *Let $h : \Gamma \longrightarrow \Gamma'$ and $k : \Gamma' \longrightarrow \Gamma''$ be morphisms of groupoids. Then h and k have simple composition and kh is a morphism from Γ to Γ'' .*

■

Examples 2.11 a) If X is a set and (Γ, m, e, s) is a groupoid then any morphism $h : X \longrightarrow \Gamma$ is equal f^T for some mapping $f : E \longrightarrow X$.

b) If G, H are groups then morphisms from G to H are just group homomorphisms.

c) If X is a set and G is a group then morphisms $X \longrightarrow G$ are points of X .

d) If $(G; A, B)$ is a double group then $m_B^T : G_A \longrightarrow G_A \times G_A$ and $m_A^T : G_B \longrightarrow G_B \times G_B$ are morphisms of groupoids. [1]

e) For any groupoid Γ the mapping $\Gamma \ni x \mapsto (e_L(x), e_R(x)) \in E \times E$ is a morphism from Γ to the pair groupoid $E \times E$. We denote this relation by \tilde{e} .

f) The relation $l : \Gamma \longrightarrow \Gamma \times \Gamma$ given by: $(x, y; z) \in Gr(l) \Leftrightarrow (x; z, y) \in Gr(m)$ is a morphism from Γ to the pair groupoid $\Gamma \times \Gamma$. It is called *left regular representation*. [1]

Remark 2.12 The above defined morphisms differ from the standard one, but later on we will see, that this definition is proper for defining the algebra of groupoid and the functorial properties of the construction. Also we want to point out that our definition *is not a generalisation* of a usual definition. Below we show that our morphisms are related to actions of groupoids on sets.

Definition 2.13 [9] Let (Γ, m, e, s) be a groupoid, Y be a set and $\mu : Y \longrightarrow \Gamma^0$ be a mapping. Denote $\Gamma \times_\mu Y := \{(x, y) \in \Gamma \times Y : e_R(x) = \mu(y)\}$.

The (left) action of Γ on Y is a mapping $\phi : \Gamma \times_\mu Y \ni (x, y) \mapsto \phi(x, y) \in Y$ which satisfy conditions:

- a) $\mu\phi(x, y) = e_L(x)$
- b) $\phi(x_1 x_2, y) = \phi(x_1, \phi(x_2, y))$ (i.e. if one side of the equality is defined the other also and are equal)
- c) $\phi(\mu(y), y) = y$

Now let Γ acts on Y . Put $f := \mu$ and $g : \Gamma \times_f Y \ni (x, y) \mapsto (\phi(x, y), y) \in Y \times Y$. Then it is easy to see that these mappings satisfy the conditions given in Prop. 2.7, so they determine morphism from Γ to the pair groupoid $Y \times Y$. Conversely, if $h : \Gamma \longrightarrow Y \times Y$ is a morphism then putting: $\mu := f_h$ and $\phi(x, y) := e'_L h_y^R(x)$ we get the action of Γ on Y . Also for any morphism $h : \Gamma \longrightarrow \Gamma'$ the mappings $\mu := f_h e'_L : \Gamma' \longrightarrow E$ and $\phi(x, x') := h_{a'}^R(x) x'$ where $a' := e'_L(x')$ define action of Γ on Γ' .

Bissections

Definition 2.14 *A bisection B is a subset of Γ such that: $e_L \mid_B : B \longrightarrow \Gamma^0$ and $e_R \mid_B : B \longrightarrow \Gamma^0$ are bijections.*

The set of bisections of Γ will be denoted by $\mathcal{B}(\Gamma)$. Bisections can also be characterized as follows:

Lemma 2.15 *A subset $B \subset \Gamma$ is a bisection $\Leftrightarrow Bs(B) = s(B)B = \Gamma^0$.*

Proof: \Rightarrow Let $x = b_1s(b_2)$ for some $b_1, b_2 \in B$. Then $e_R(b_2) = e_R(b_1)$ and, since B is a bisection, $b_2 = b_1$ and $x \in \Gamma^0$, so $Bs(B) \subset \Gamma^0$. Moreover for any $x \in \Gamma^0$ we can find $b \in B$ with $x = e_L(b) = bs(b)$ so $\Gamma^0 \subset Bs(B)$ and $Bs(B) = \Gamma^0$. In the same way we have $s(B)B = \Gamma^0$.

\Leftarrow Suppose that for $b_1, b_2 \in B$ we have $e_R(b_1) = e_R(b_2)$. Then b_1 and $s(b_2)$ are composable, so $b_1s(b_2) \in \Gamma^0$ and $b_1 = b_2$. The same holds for the left projection. So $e_L|_B$ and $e_R|_B$ are injective. But for any $x \in \Gamma^0$ we can find $b_1, b_2 \in B$ with $x = e_L(b_1) = e_R(b_2)$.

■

For a bisection B let $Bx := B\{x\}$ and $xB := \{x\}B$. The following proposition is easy to prove:

Proposition 2.16 *Let B be a bisection.*

- a) $L_B : \Gamma \ni x \mapsto Bx \in \Gamma$ and $R_B : \Gamma \ni x \mapsto xB \in \Gamma$ are bijections.
- b) $L_B (R_B)$ preserves right (left) fibers.
- c) $L_B(F_l(x)) = F_l(Bx)$, $R_B(F_r(x)) = F_r(xB)$.
- d) $B(xy) = (Bx)y$, $(xy)B = x(yB)$.
- e) If $B, C \in \mathcal{B}(\Gamma)$ then $BC \in \mathcal{B}(\Gamma)$ and $\mathcal{B}(\Gamma)$ is a group.
- f) $L_B (R_B)$ is a left (right) action of $\mathcal{B}(\Gamma)$ on Γ .

■

Examples 2.17 a) For any groupoid the set of identities is a bisection.

b) If Γ is a group then bisections are just group elements.

c) If $\Gamma := X \times X$ is a pair groupoid then any bisection is of the form $B := \{(f(x), x) : x \in X\}$ for some bijection $f : X \rightarrow X$.

d) If $(G; A, B)$ is a double group then for any $b \in B$ the sets bA, Ab are bisections of G_A , the mapping $B \ni b \mapsto bA \in \mathcal{B}(\Gamma)$ is a group homomorphism.

Proposition 2.18 *Let $h : \Gamma \rightarrow \Gamma'$ be a morphism and B a bisection of Γ then the set $h(B)$ is a bisection of Γ' .*

Proof: Let us take any $a' \in E'$ and let $E \ni a := f_h(a')$. Then there exist unique points $x, y \in B$ with $e_L(x) = a = e_R(y)$, so there exist unique $x', y' \in \Gamma'$ such that $(x', x) \in Gr(h)$, $e'_L(x') = a'$ and $(y', y) \in Gr(h)$, $e'_R(y') = a'$.

■

Remark 2.19 Since bisections defines bijections of Γ they acts, in this purely algebraic context, on $\mathcal{A}(\Gamma)$. This action commutes with right multiplication. Since morphisms acts also on bisections we can expect that they are unitary multipliers on $C^*(\Gamma)$. And this is true. Later on we will see that also in the differential setting bisections can be interpreted as multipliers.

Remark 2.20 One can think of groupoids as of some generalisation of groups and treat groupoid elements in the same way as group elements. But this analogy can be misleading since for groups bisections are just elements. So group elements have some properties “because they are groupoid elements” and others “because they are bisections”.

3 Differential groupoids

From now on, when we use the word *manifold* without any comments, we mean Hausdorff, finite dimensional, smooth manifold with a countable basis of neighbourhoods. *Submanifold* is a nonempty, embedded submanifold (with the relative topology).

A *differentiable relation* $r : X \rightarrow Y$ is a triple $r = (R; Y, X)$ such that X, Y are manifolds and R is a submanifold in $Y \times X$.

If $r = (R; Y, X)$ is a differentiable relation then its *tangent lift* is a relation $Tr : TX \longrightarrow TY$ with a graph $Gr(Tr) := TGr(r)$. A *phase lift* of r is a relation $Pr : T^*X \longrightarrow T^*Y$ such that:

$$(\alpha, \beta) \in Gr(Pr) \iff \langle \alpha, u \rangle = \langle \beta, v \rangle \text{ for any } (u, v) \in T_{(y,x)}Gr(r),$$

where $y := \pi_Y(\alpha)$, $x := \pi_X(\beta)$ and π_X, π_Y are the canonical projections in the cotangent bundles.

We say that relations $r : X \longrightarrow Y$ and $s : Y \longrightarrow Z$ are *transverse* iff Tr, Ts and Pr, Ps have simple composition, and sr is a differentiable relation. Such situation will be denoted by $r \uparrow s$.

Let us also recall that a *differentiable reduction* is a differentiable relation $r : X \longrightarrow Y$ of the form $r = fi^T$, where $i : C \longrightarrow X$ is an inclusion map of the submanifold $C \subset X$ and $f : C \longrightarrow Y$ is a surjective submersion.

Definition 3.1 [2] A *differential groupoid* (Γ, m, e, s) is a groupoid such that Γ is a manifold, m is a differentiable reduction, e is a differentiable relation, s is a diffeomorphism and the following transversality conditions hold: $m \uparrow (m \times id)$, $m \uparrow (id \times m)$, $m \uparrow (e \times id)$, $m \uparrow (id \times e)$.

It can be shown [2] that in this situation e_L, e_R are submersions.

Examples 3.2 a) Examples 2.4 a)-e) after obvious smoothness conditions are differential groupoids.

b) *Double Lie groups*. We say that a double group $(G; A, B)$ is a double Lie group iff G is a Lie group and A, B are closed subgroups of G . Then G_A, G_B are differential groupoids.

c) *Tangent and cotangent bundles*. If X is a manifold then $(TX, +, X, -)$ and $(T^*X, +, X, -)$ are differential groupoids. More generally, if (P, X) is a vector bundle then it is a groupoid in a natural way $(P, +, X, -)$.

d) *Tangent and phase lifts of differential groupoids* [2]. If (Γ, m, e, s) is a differential groupoid then $(T\Gamma, Tm, Te, Ts)$ and $(T^*\Gamma, Pm, Pe, -Ps)$ are differential groupoids. If $\Gamma := (X, d^T, X, id)$ is a manifold groupoid then its tangent lift $T\Gamma = (TX, d_{TX}^T, TX, id)$ is again manifold groupoid but its cotangent lift $P\Gamma = (T^*X, +, X, -)$ is a cotangent bundle with usual groupoid structure.

e) If $\Gamma = (G, m, e, s)$ is a Lie group, then its tangent lift is a Lie group TG . But the phase lift is T^*G as a transformation groupoid: $T^*G = G \times \mathfrak{g}^*$ with a coadjoint action.

Morphisms of differential groupoids.

Definition 3.3 [2] Let Γ, Γ' be differential groupoids and $h : \Gamma \longrightarrow \Gamma'$ a differentiable relation which is a (algebraic) morphism of groupoids. Then h is a morphism of differential groupoids iff $m' \uparrow (h \times h)$ and $h \uparrow e$.

Proposition 3.4 [2] If $h : \Gamma \longrightarrow \Gamma'$ is a morphism of differential groupoids and $f_h := h_0^T$ then:

- a) $f_h : E' \longrightarrow E$ is a smooth mapping.
- b) $\Gamma *_h E' := \{(x, b) \in \Gamma \times E' : e_L(x) = f_h(b)\}$ is a submanifold of $\Gamma \times E'$ (and of $\Gamma \times \Gamma'$).
- c) The mapping: $Gr(h) \ni (y, x) \mapsto (x, e'_L(y)) \in \Gamma *_h E'$ is a diffeomorphism.

■

In the next lemma we collect the properties of various sets and mappings associated with morphisms of differential groupoids, which will be used later on.

Lemma 3.5 Let $h : \Gamma \longrightarrow \Gamma'$ be a morphism of differential groupoids.

- a) $\Gamma *_h \Gamma' := \{(x, y) \in \Gamma \times \Gamma' : e_L(x) = f_h(e'_L(y))\}$ is a submanifold of $\Gamma \times \Gamma'$.
- b) $\Gamma \times_h E' := \{(x, b) \in \Gamma \times E' : e_R(x) = f_h(b)\}$ is a submanifold of $\Gamma \times E'$.
- c) $\Gamma \times_h \Gamma' := \{(x, y) \in \Gamma \times \Gamma' : e_R(x) = f_h(e'_L(y))\}$ is a submanifold of $\Gamma \times \Gamma'$.
- d) Let $(x, y) \in \Gamma \times_h \Gamma'$ and $b := e'_L(y)$, then the mappings:
 $m_h : \Gamma \times_h \Gamma' \ni (x, y) \mapsto m'(h_b^R(x), y) \in \Gamma'$ and $\pi_2 : \Gamma \times_h \Gamma' \ni (x, y) \mapsto y \in \Gamma'$
are surjective submersions.
- e) The mapping $t_h : \Gamma \times_h \Gamma' \ni (x, y) \mapsto (x, m_h(x, y)) \in \Gamma *_h \Gamma'$ is a diffeomorphism.

- f) For $b \in E'$, $h_b^R : \Gamma \supset F_r(f(b)) \longrightarrow F_r(b) \subset \Gamma'$ and $h_b^L : \Gamma \supset F_l(f(b)) \longrightarrow F_l(b) \subset \Gamma'$ are smooth mappings.
- g) $\tilde{\pi}_2 : \Gamma *_h \Gamma' \ni (x, y) \mapsto y \in \Gamma'$ is surjective submersion.
- h) The sets $\Gamma \times_h \Gamma'_a, \Gamma *_h \Gamma'_a$ are submanifolds of $\Gamma \times_h \Gamma'$ and $\Gamma *_h \Gamma'$. Also statements d), e), g) remains true after suitable restriction of the corresponding mappings.

Proof:

- a) Consider $id \times e'_L : \Gamma \times \Gamma' \longrightarrow \Gamma \times E'$ - this is smooth submersion and $\Gamma *_h \Gamma' = (id \times e'_L)^{-1}(\Gamma *_h E')$. The assertion follows from second item of the previous proposition.
- b) $\Gamma \times_h E' = (s \times id)(\Gamma *_h E')$ but $(s \times id)$ is a diffeomorphism so again we use second item of the previous proposition.
- c) Write $\Gamma \times_h \Gamma' = (id \times e'_L)^{-1}(\Gamma \times_h E')$ and use b).
- d) From the third statement of the previous proposition we know that the mapping $h^L : \Gamma *_h E' \ni (x, b) \mapsto h_b^L(x) \in \Gamma'$ is smooth. Now the mapping $h^R : \Gamma \times_h E' \ni (x, b) \mapsto h_b^R(x) \in \Gamma'$ is the composition of : $s'h^L(s \times id) : \Gamma \times_h E' \longrightarrow \Gamma'$ so it is smooth. Now m_h is the composition:
- $$m_h : \Gamma \times_h \Gamma' \xrightarrow{id \times e'_L \times id} \Gamma \times E' \times \Gamma' \xrightarrow{h^R \times id} \Gamma'(2) \xrightarrow{m'} \Gamma'$$
- $$(x, y) \mapsto (x, e'_L(y), y) \mapsto (h_{e'_L(y)}^R(x), y) \mapsto m'(h_{e'_L(y)}^R(x), y)$$
- and is a smooth mapping. Since for any $y \in \Gamma'$ $m_h(f_h(e'_L(y)), y) = y$ it is clear that m_h is surjective. The mapping m_h is illustrated on the picture below.

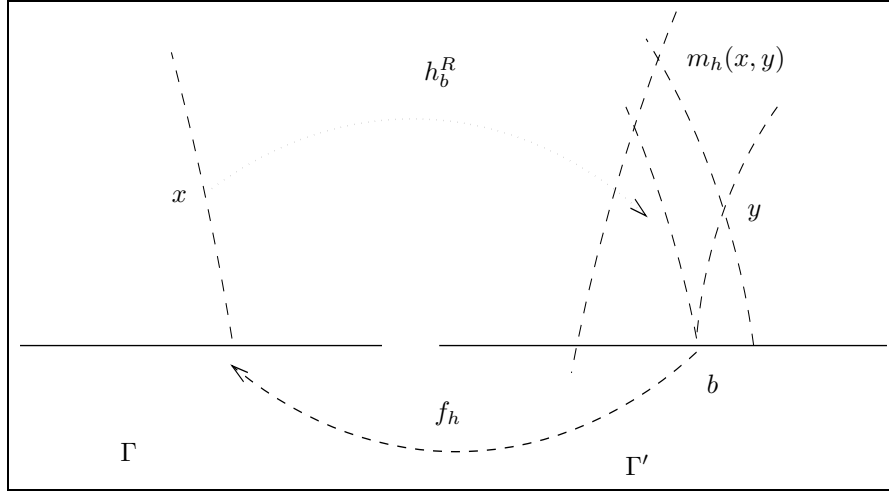


Figure 3.1:

Now let $m_h(x_0, y_0) = z_0$ and let $z(t), z(0) = z_0$ be a curve through z_0 . Then $f_h(e'_L(z(t)))$ is a curve in E through $e_L(x_0)$. It can be lifted to $x(t)$ - a curve through x_0 with $e_L(x(t)) = f_h(e'_L(z(t)))$. Since h^L is smooth $w(t) := h^L(x(t), f_h(e'_L(z(t))))$ is a curve in Γ' with $e'_L(w(t)) = e'_L(z(t))$ and $y(t) := m'(s'(w(t)), z(t))$ is a curve through y_0 . One can check that $m_h(x(t), y(t)) = z(t)$. The statement about π_2 is obvious.

- e) From the previous statement t_h is smooth and open. It is clear that it is an immersion. So it is enough to show that it is surjective. Let $(x, z) \in \Gamma *_h \Gamma'$ with $b := e'_L(z)$. Then it is easy to see that $(x, z) = t_h(x, s'(h_b^L(x))z)$.
- f) It follows from the fact that h^L, h^R are smooth.
- g) It is obvious.
- h) The proofs are simmilar to the proofs of points d), e) and g).

■

Examples 3.6 a) If $f : X \rightarrow Y$ is a smooth mapping then T^*f considered as a relation: $T^*Y \rightarrow T^*X$ is a morphism of differential groupoids. The same is true for $(Tf)^T : TY \rightarrow TX$. Note that here TX, TY are considered as manifold groupoids, not vector bundle groupoids. Unless f is a local diffeomorphism, $(Tf)^T$ is not a morphism of $(TY, +, Y, -)$ and $(TX, +, X, -)$.

b) Let a Lie group G acts on a manifold X . We form a transformation groupoid $\Gamma := G \times X$. Let Y be a manifold and $h : \Gamma \rightarrow Y \times Y$ be a morphism to the pair groupoid of Y . Consider the smooth mapping $\Phi : G \times Y \ni (g, y) \mapsto e_L^R h(g, f_h(y); y) \in Y$. Then Φ defines an action of G on Y . Moreover f_h is equivariant i.e. $gf_h(y) = f_h\Phi(g, y)$. Conversely, if G acts on X and Y with equivariant mapping $f : Y \rightarrow X$ then h defined by: $Gr(h) := \{(gy, y; g, f(y)) : y \in Y, g \in G\}$ is a morphism from Γ to $Y \times Y$.

c) Let X, Y be manifolds and $\Gamma := X \times X, \Gamma' := Y \times Y$ be corresponding pair groupoids. Then using Props. 2.7 and 3.4 one can see that any morphism $h : \Gamma \rightarrow \Gamma'$ is determined by a smooth surjection $f : Y \rightarrow X$ and a smooth mapping $g : X \times Y \rightarrow Y$ which satisfy for any $x, x_1 \in X, y \in Y$ conditions:

a) $fg(x, y) = x$, b) $g(f(y), g(x, y)) = y$, c) $g(x, y) = g(x, g(x_1, y))$.

Then $Gr(h) := \{(g(x, y), y; x, f(y)) : x \in X, y \in Y\}$. From b) it follows that g is a surjection. From c) and b): $g(f(y), y) = g(f(y), g(x, y)) = y$. If $x_0 = f(y_0)$ and $x(t)$ is a curve through x_0 then $g(x(t), y_0)$ is a curve through y_0 . Since $x(t) = fg(x(t), y_0) - f$ is a submersion. Choose some $x_0 \in X$ and let $Z := f^{-1}(x_0)$ —this is a submanifold of Y . We claim that the mapping: $\phi : X \times Z \ni (x, y) \mapsto g(x, y) \in Y$ is a diffeomorphism. It is clear that this mapping is smooth. If $g(x_1, y_1) = g(x_2, y_2)$ then from a) $x_1 = x_2$. So $y_1 = g(f(y_1), g(x, y_1)) = g(f(y_2), g(x, y_2)) = y_2$ and ϕ is an injection. For $y \in Y$ we have: $y = g(f(y), g(x_0, y))$ but $g(x_0, y) \in Z$ so it is a surjection. From a): $g(\dot{x}, \dot{y}) = 0 \Rightarrow \dot{x} = 0$. If $y(t)$ is a curve in Z through y then $y(t) = g(x_0, y(t)) = g(x_0, g(x, y(t)))$. From this follows that ϕ is an immersion. So it is a diffeomorphism. In this way for any morphism $h : \Gamma \rightarrow \Gamma'$ there exists diffeomorphism $\phi : Y \rightarrow X \times Z$ and $Gr((\phi \times \phi)h) = \{(x, z, x_1, z; x, x_1) : x, x_1 \in X, z \in Z\}$.

The next proposition shows that differential groupoids with the above defined morphisms form a category.

Proposition 3.7 [2] *Let $\Gamma, \Gamma', \Gamma''$ be differential groupoids and let $h : \Gamma \rightarrow \Gamma'$ and $k : \Gamma' \rightarrow \Gamma''$ be morphisms. Then $h \dashv k$ and $kh : \Gamma \rightarrow \Gamma''$ is a morphism.*

■

A submanifold $B \subset \Gamma$ is a *bissection* iff $e_L|_B$ and $e_R|_B$ are diffeomorphisms. If $h : \Gamma \rightarrow \Gamma'$ is a morphism of differential groupoids and B is a bissection then $h(B)$ is the image of E' by the mapping:

$$E' \ni a' \mapsto (f_h(a'), a') \mapsto ((e_R|_B)^{-1} f_h(a'), a') \mapsto h^R((e_R|_B)^{-1} f_h(a'), a') \in \Gamma'$$

also $h(B)$ is the image of E' by the mapping:

$$E' \ni a' \mapsto (f_h(a'), a') \mapsto ((e_L|_B)^{-1} f_h(a'), a') \mapsto h^L((e_L|_B)^{-1} f_h(a'), a') \in \Gamma'.$$

These mappings are smooth sections of the projections e_R' and e_L' respectively. So $h(B)$ is a bissection of Γ' .

4 A *-algebra of a differential groupoid.

In this section we introduce a *-algebra of a differential groupoid. The way we do it is rather non-standard and at the first look may be regarded as too complicated, nevertheless it is very convenient for the further development.

Let Γ be a differential groupoid and let $\Omega^{1/2}(e_L), (\Omega^{1/2}(e_R))$ be the smooth bundle of complex, half densities along the left (right) fibers of Γ . Following A. Connes [4] our basic object is the linear space of compactly supported smooth sections of the bundle $\Omega^{1/2}(e_L) \otimes \Omega^{1/2}(e_R)$. We denote this space by $\mathcal{A}(\Gamma)$. Its elements will be called *bidensities* and usually denoted by ω . So $\omega(x) = \lambda(x) \otimes \rho(x) \in \Omega^{1/2}T_x^l \Gamma \otimes \Omega^{1/2}T_x^r \Gamma$, where we used notation: $T_x^l \Gamma := T_x(F_l(x)), T_x^r \Gamma := T_x(F_r(x))$. In the following we also write $\Omega_L^{1/2}(x) := \Omega^{1/2}T_x^l \Gamma$ and $\Omega_R^{1/2}(x) := \Omega^{1/2}T_x^r \Gamma$.

We also use the following notation: if M, N are manifolds, $F : M \rightarrow N$ and Ψ is some geometric object on

M which can be pushed-forward by F , then we denote the push-forward of ψ simply by $F\psi$. What it really means will be clear from the context.

The groupoid inverse induces the star operation on \mathcal{A} as follows

$$\omega^*(x)(v \otimes w) := \overline{\omega(s(x))(s(w) \otimes s(v))}, \quad v \in \Lambda^{max} T_x^l \Gamma, \quad w \in \Lambda^{max} T_x^r \Gamma$$

(for any vector space V by $\Lambda^{max} V$ we denote the maximal non zero exterior power of V). This is well defined antylinear involution. (since s is an involutive diffeomorphism which interchanges left and right fibers).

We are going to equip \mathcal{A} with multiplication, which gives it a $*$ -algebra structure. First we will show that with any morphism $h : \Gamma \rightrightarrows \Gamma'$ is associated a mapping $\hat{h} : \mathcal{A}(\Gamma) \longrightarrow L\mathcal{A}(\Gamma')$ ($L\mathcal{A}(\Gamma')$ denote linear endomorphisms of $\mathcal{A}(\Gamma')$), which “well behave” with respect to composition of morphisms and $*$ -operation. Then putting $h = id$ we get algebra structure on $\mathcal{A}(\Gamma)$ and \hat{h} will became $*$ -algebra homomorphism from $\mathcal{A}(\Gamma)$ to the left algebraic multipliers of $\mathcal{A}(\Gamma')$. Before this we define some special sections of $\Omega^{1/2}(e_L) \otimes \Omega^{1/2}(e_R)$ which are very convenient for computations.

***-invariant bidensities.**

Since left(right) translations are diffeomorphisms of left(right) fibers, we can define left(right) invariant sections of $\Omega^{1/2}(e_L)(\Omega^{1/2}(e_R))$, namely a section λ is left invariant iff for any $(x, y) \in \Gamma^{(2)}$ $\lambda(xy)(xv) = \lambda(y)(v)$ $v \in \Lambda^{max} T_y^l \Gamma$. In the same way are defined right invariant half densities. Any left invariant half density is determined by its value on Γ^0 and conversely any section of $\Omega^{1/2}(e_L)|_{\Gamma^0}$ can be uniquely extended to left invariant half density on Γ .

So let $\tilde{\lambda}$ be a non-vanishing, real, half density on Γ^0 along the left fibers. (one constructs such a density by covering Γ^0 with maps submitted to submersion e_L and using appropriate partition of unity to glue them together) We define:

$$\lambda_0(x)(v) := \tilde{\lambda}(e_R(x))(s(x)v), \quad v \in \Lambda^{max} T_x^l \Gamma,$$

then λ_0 is a left invariant, non vanishing section of $\Omega^{1/2}(e_L)$. Now $\tilde{\rho} := \tilde{\lambda}s$ is non vanishing, real, half density on Γ^0 along the right fibers, and ρ_0 defined by: $\rho_0(x)(v) := \tilde{\rho}(e_L(x)(vs(x)))$, $v \in \Lambda^{max} T_x^r \Gamma$ is a right invariant, non vanishing, real half density along the right fibers. Let $\omega_0 := \lambda_0 \otimes \rho_0$, then this is real, non vanishing bidensity.

From now on the symbol ω_0 will always mean bidensity constructed in this way. When ω_0 is choosen any element $\omega \in \mathcal{A}(\Gamma)$ can be written uniquely as $\omega = f\omega_0$ for some smooth, complex function f with compact support. Note the following:

Lemma 4.1 *If $\omega = f\omega_0$ then $\omega^* = f^*\omega_0$ where $f^*(x) := \overline{f(s(x))}$.*

Proof: $\omega^*(x)(v \otimes w) := \overline{\omega(s(x))(s(w) \otimes s(v))}$, $v \in \Lambda^{max} T_x^l \Gamma$, $w \in \Lambda^{max} T_x^r \Gamma$.

$$\begin{aligned} \overline{\omega(s(x))(s(w) \otimes s(v))} &= \overline{f(s(x))(\lambda_0(s(x)) \otimes \rho_0(s(x)))(s(w) \otimes s(v))} = \\ &= f^*(x)\lambda_0(e_L(x))(xs(w))\rho_0(e_R(x))(s(v)x) = f^*(x)\rho_0(e_L(x))(ws(x))\lambda_0(e_R(x))(s(x)v) = \\ &= f^*(x)\lambda_0(x)(v)\rho_0(x)(w) = f^*(x)\omega_0(x)(v \otimes w). \end{aligned}$$

■

Remark 4.2 Choosing λ_0 in fact we choose some left Haar system in the sense of [3] on our groupoid. But all our constructions and in particular our C^* algebra are independent of this choice.

Action of groupoid morphisms on bidensities.

Now we are going to construct for a morphism of diffrential groupoids h the mapping \hat{h} .

Let $h : \Gamma \rightrightarrows \Gamma'$ be a morphism of differential groupoids. Then from lemma 3.5 we know that:

1. The set $\Gamma \times_h \Gamma'_a := \{(x, y) \in \Gamma \times \Gamma' : e_R(x) = f_h(e'_L(y)), e_R(y) = a\}$ is a submanifold of $\Gamma \times_h \Gamma'$.
2. The mapping: $\pi_2 : \Gamma \times_h \Gamma'_a \ni (x, y) \mapsto y \in \Gamma'_a$ is a surjective submersion and $\pi_2^{-1}(y)$ is diffeomorphic to $F_r(f_h(e'_L(y)))$.
3. The mapping

$$t_h : \Gamma \times_h \Gamma'_a \ni (x, y) \mapsto (x, m_h(x, y)) \in \Gamma *_h \Gamma'_a := \{(x, y) \in \Gamma \times \Gamma'_a : e_L(x) = f_h(e'_L(y))\}$$

is a diffeomorphism.

4. $\tilde{\pi}_2 : \Gamma *_h \Gamma'_a \ni (x, y) \mapsto y \in \Gamma'_a$ is a surjective submersion and $\tilde{\pi}_2^{-1}(y)$ is diffeomorphic to $F_l(f_h(e'_L(y)))$.

Before we go further, let us recall some facts about densities. Let V be a finite dimensional vector space. For $p \geq 0$ we denote the linear space of complex p -densities on V by $\Omega^p(V)$. If $V = V_1 \oplus V_2$ and ν_1, ν_2 are p -densities on V_1, V_2 then the formula $(\nu_1 \otimes \nu_2)(v_1 \wedge v_2) := \nu_1(v_1)\nu_2(v_2)$ for $v_1 \in \Lambda^{max} V_1, v_2 \in \Lambda^{max} V_2$ defines isomorphism $\Omega^p(V) = \Omega^p(V_1) \otimes \Omega^p(V_2)$. Also we have $\Omega^p(V) = \Omega^p(V_1) \otimes \Omega^p(V/V_1)$ defined by choosing some $V_2 \subset V$ complementary to V_1 . The isomorphism does not depend from the choice made. In this way if $F : V \longrightarrow W$ is a linear surjection, we have canonical isomorphism $\Omega^p(V) = \Omega^p(\ker F) \otimes \Omega^p(W)$. This fact is constantly used in the following. Now we go back to groupoid morphisms.

Let $(x, y) \in \Gamma \times_h \Gamma'_a$ and $\mathbf{t}_h(x, y) =: (x, z), b := e'_L(z)$. Due to the point 2. we have isomorphism: $i_1 : \Omega_R^{1/2}(x) \otimes \Omega_R^{1/2}(y) \longrightarrow \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_a)$. From point 3) $\mathbf{t}_h : \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_a) \longrightarrow \Omega^{1/2} T_{(x,z)}(\Gamma *_h \Gamma'_a)$ is an isomorphism and from 4) $i_2 : \Omega_L^{1/2}(x) \otimes \Omega_R^{1/2}(z) \longrightarrow \Omega^{1/2} T_{(x,z)}(\Gamma *_h \Gamma'_a)$ is an isomorphism. In this way we get equality: $(i_2)^{-1} \mathbf{t}_h i_1(\rho_x \otimes \rho_y) =: \lambda_x \otimes \rho_z$ for some $\lambda_x \otimes \rho_z \in \Omega_L^{1/2}(x) \otimes \Omega_R^{1/2}(z)$. Moreover the mapping $F_l(y) \ni u \mapsto h_{e'_L(y)}^R(x)u \in F_l(z)$ is a diffeomorphism, so for $\lambda_y \in \Omega_L^{1/2}(y)$, we have $h_{e'_L(y)}^R(x)\lambda_y \in \Omega_L^{1/2}(z)$. Now let $\omega = \lambda \otimes \rho \in \mathcal{A}(\Gamma)$ $\omega' = \lambda' \otimes \rho' \in \mathcal{A}(\Gamma')$. Then: $(i_2)^{-1} \mathbf{t}_h i_1(\rho(x) \otimes \rho'(y)) =: \tilde{\lambda}_x \otimes \tilde{\rho}_z$ and $h_{e'_L(y)}^R(x)\lambda'(y) =: \tilde{\lambda}'_z$. So the expression: $[\lambda(x)\tilde{\lambda}_x] \otimes \tilde{\lambda}'_z \otimes \tilde{\rho}_z$ defines one-density on $F_l(f_h(b))$ with values in one dimensional vector space $\Omega_L^{1/2}(z) \otimes \Omega_R^{1/2}(z)$.

Let us define

$$(\hat{h}(\omega)\omega')(z) := \int_{F_l(f_h(b))} [\lambda\tilde{\lambda}] \otimes \tilde{\lambda}'_z \otimes \tilde{\rho}_z.$$

Choose: $\omega_0 = \lambda_0 \otimes \rho_0, \omega'_0 = \lambda'_0 \otimes \rho'_0$. Then $\omega = f_1 \omega_0, \omega' = f_2 \omega'_0$ and $(i_2)^{-1} \mathbf{t}_h i_1(\rho_0(x) \otimes \rho'_0(y)) =: t_h(x, y)\lambda_0(x) \otimes \rho'_0(z)$ for some smooth, nonvanishing function $t_h : \Gamma \times_h \Gamma' \longrightarrow R$ and $h_{e'_L(y)}^R(x)\lambda'_0(y) = \lambda'_0(z)$. We get the explicit expression:

$$(\hat{h}(\omega)\omega')(z) := \left[\int_{F_l(f_h(b))} \lambda_0^2(x) f_1(x) t_h(x, y) f_2(y) \right] \omega'_0(z) =: (f_1 *_h f_2)(z) \omega'_0(z),$$

where y is defined by $\mathbf{t}_h(x, y) = (x, z)$, i.e. $y = s'(h_b^L(x))z$.

The next proposition is crucial for the construction, it describes how the mapping \hat{h} behaves with respect to a composition of morphisms. We cannot simply write: $\hat{k}(\hat{h}\omega) = \hat{k}h\omega$ since the left hand side is not defined. Instead of this equality we prove the other one, which is formally the same as for morphism of C^* -algebras. (see Appendix E)

Proposition 4.3 *Let $h : \Gamma \longrightarrow \Gamma', k : \Gamma' \longrightarrow \Gamma''$ be morphisms of differential groupoids. Then*

$$\hat{k}(\hat{h}(\omega_1)\omega_2)\omega_3 = \hat{k}h(\omega_1)(\hat{k}(\omega_2)\omega_3) \text{ for any } \omega_1 \in \mathcal{A}(\Gamma), \omega_2 \in \mathcal{A}(\Gamma'), \omega_3 \in \mathcal{A}(\Gamma'').$$

Proof: Choose $\omega_0 := \lambda_0 \otimes \rho_0, \omega'_0 := \lambda'_0 \otimes \rho'_0, \omega''_0 := \lambda''_0 \otimes \rho''_0$ and write $\omega_1 = f_1 \lambda_0 \otimes \rho_0, \omega_2 = f_2 \lambda'_0 \otimes \rho'_0, \omega_3 = f_3 \lambda''_0 \otimes \rho''_0$.

Let $z \in \Gamma'', a := e''_L(z), b := f_k(a), c := f_h(b) = f_h f_k(a) = f_{kh}(a)$.

Compute the left hand side of the equality:

$(\hat{k}(\hat{h}(\omega_1)\omega_2)\omega_3)(z) = ((f_1 *_h f_2) *_k f_3)(z) \omega''(z)$ and

$$\begin{aligned} ((f_1 *_h f_2) *_k f_3)(z) &= \int_{F_l(b)} \lambda_0^2(y) (f_1 *_h f_2)(y) t_k(y, z_1) f_3(z_1) = \\ &= \int_{F_l(b)} \lambda_0^2(y) \left[\int_{F_l(c)} \lambda_0^2(x) f_1(x) t_h(x, y_1) f_2(y_1) \right] t_k(y, z_1) f_3(z_1) = \\ &= \int_{F_l(b) \times F_l(c)} (\lambda_0^2(y) \otimes \lambda_0^2(x)) [f_1(x) f_2(y_1) f_3(z_1) t_h(x, y_1) t_k(y, z_1)], \end{aligned}$$

where y_1, z_1 are given by: $\mathbf{t}_h(x, y_1) = (x, y), \mathbf{t}_k(y, z_1) = (y, z)$.

The situation is illustrated on the figure 4.2.

The right hand side: $(\hat{k}h(\omega_1)(\hat{k}(\omega_2)\omega_3))(z) = (f_1 *_k h(f_2 *_k f_3))(z) \omega''(z)$ and

$$(f_1 *_k h(f_2 *_k f_3))(z) = \int_{F_l(c)} \lambda_0^2(x) f_1(x) t_{kh}(x, z_2) (f_2 *_k f_3)(z_2) =$$

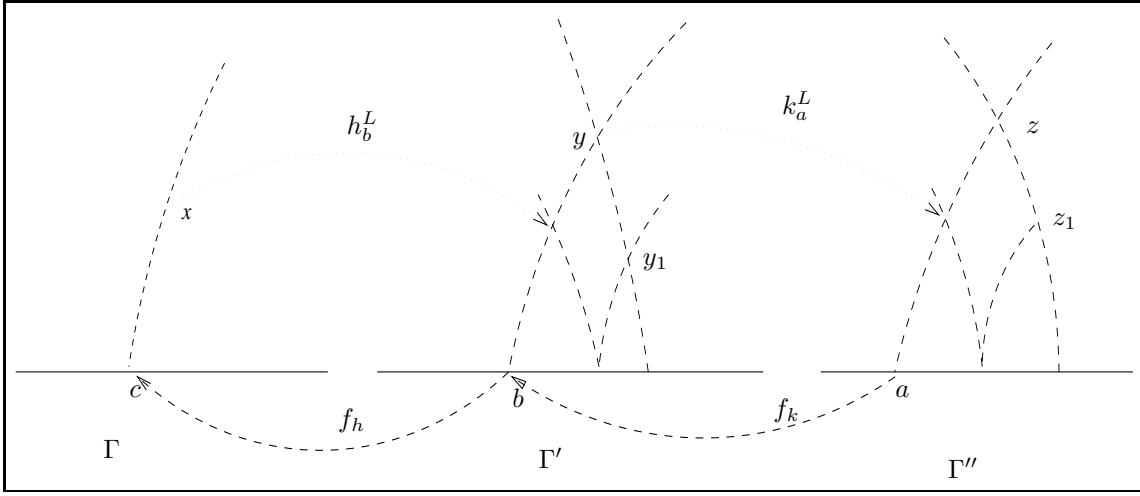


Figure 4.2:

$$= \int_{F_l(c)} \lambda_0^2(x) f_1(x) t_{kh}(x, z_2) \int_{F_l(b_2)} \lambda_0^2(y_2) f_2(y_2) t_k(y_2, z_3) f_3(z_3),$$

where y_2, z_2, z_3 are given by: $t_{kh}(x, z_2) = (x, z)$, $t_k(y_2, z_3) = (y_2, z_2)$ and $b_2 := f_k(e_L''(z_2))$.

For fixed $x \in F_l(c)$ the mapping: $F_l(b_2) \ni y_2 \mapsto m_h(x, y_2) \in F_l(b)$ is a diffeomorphism. Using this fact, above integral is equal:

$$\int_{F_l(b) \times F_l(c)} (\lambda_0^2(y) \otimes \lambda_0^2(x)) [f_1(x) f_2(y_2) f_3(z_3) t_{kh}(x, z_2) t_k(y_2, z_3)],$$

where y_2, z_2, z_3 are defined by: $t_{kh}(x, z_2) = (x, z)$, $t_h(x, y_2) = (x, y)$, $t_k(y_2, z_3) = (y_2, z_2)$.

The situation is illustrated on the figure 4.3.

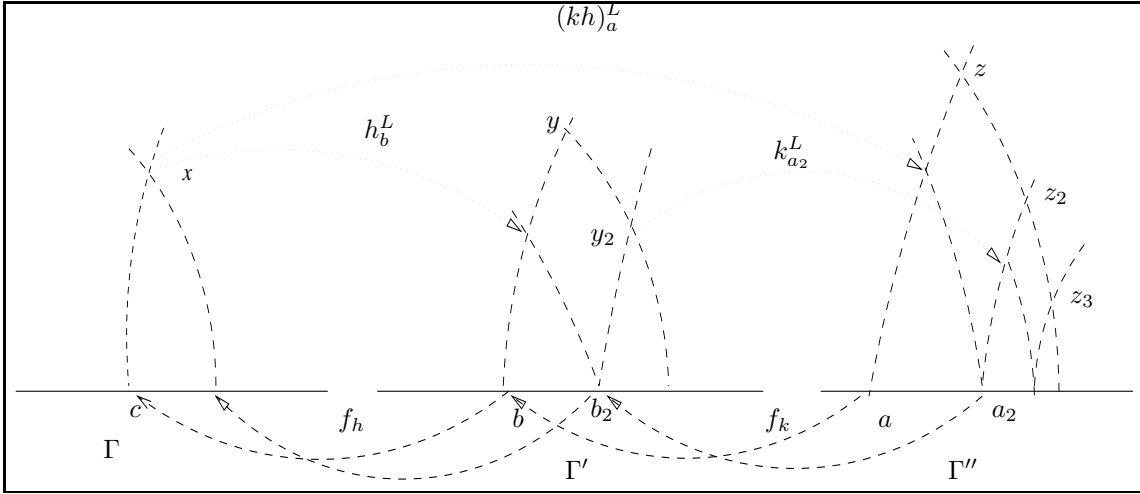


Figure 4.3:

Comparing this expression with the left hand side, we see that $y_2 = y_1$.

We prove that $z_3 = z_1$. We have: $z_3 = s''(k_{a_2}^L(y_2))z_2$, $y_2 = s'(h_b^L(x))y$ and $z_2 = s''((kh)_a^L(x))z$. So $z_3 = s''(\tilde{z})z$ for $\tilde{z} := (kh)_a^L(x) k_{a_2}^L(y_2) = (kh)_a^L(x) k_{a_2}^L[s'(h_b^L(x))y]$ and $z_1 = s''(k_a^L(y))z$. We have $e_L''(\tilde{z}) = e_L''(z) = a$ and $e_L''(k_a^L(y)) = a$ so it is enough to show that $(\tilde{z}, y) \in Gr(k)$.

Since $((kh)_a^L(x), x) \in Gr(kh)$ there exists unique (composition of morphisms is simple!) \tilde{y} satisfying $((kh)_a^L(x), \tilde{y}) \in Gr(k)$ and $(\tilde{y}, x) \in Gr(h)$ but then $e_L'(\tilde{y}) = f_k(a) = b$ so $\tilde{y} = h_b^L(x)$. Then $(\tilde{z}; h_b^L(x), y_2) \in Gr(m''(k \times k) = Gr(km'))$ and $(\tilde{z}, h_b^L(x)y_2) \in Gr(k)$. But $h_b^L(x)y_2 = h_b^L(x)s'(h_b^L(x))y = y$ and $(\tilde{z}, y) \in Gr(k)$. So to prove equality it remains to show that: $t_h(x, y_1) t_k(y, z_1) = t_{kh}(x, z_2) t_k(y_1, z_1)$. This is done in the following:

Lemma 4.4 Let $(x, y, z) \in \Gamma \times \Gamma' \times \Gamma''$ satisfy: $e_R(x) = f_h(e'_L(y))$, $e'_R(y) = f_k(e''_L(z))$ and let y', z' be defined by: $t_h(x, y) = (x, y')$ and $t_k(y, z) = (y, z')$. Then

$$t_h(x, y)t_k(y', z) = t_{kh}(x, z')t_k(y, z).$$

Proof: The proof is given in the end of this section.

■

Corollary 4.5 The algebra structure on $\mathcal{A}(\Gamma)$. Take $h = id : \Gamma \rightarrow \Gamma$, then $\Gamma \times_h \Gamma = \Gamma^{(2)}$ and put: $\omega_1 \omega_2 := \hat{id}(\omega_1) \omega_2$. Due to the above proposition, this product is associative. Chosen ω_0 we can write $\omega_1 = f_1 \omega_0$, $\omega_2 = f_2 \omega_0$. Let us show that in this situation $t_{id} \equiv 1$.

We have $\Gamma \times_{id} \Gamma_c = \{(x, y) \in \Gamma \times \Gamma : e_R(x) = e_L(y), e_R(y) = c\}$, $\Gamma *_{id} \Gamma_c = \{(x, y) \in \Gamma \times \Gamma : e_L(x) = e_L(y), e_R(y) = c\}$ and $t_{id}(x, y) = (x, xy)$. Let $(x, y) \in \Gamma \times_{id} \Gamma_c$, $z := xy$, $e_R(x) = b$, $e_L(x) = e_R(y) = a$, $V := T_{(x, y)} \Gamma \times_{id} \Gamma_c$, $W := T_{(x, z)} \Gamma *_{id} \Gamma_c$. It is easy to see that the following decompositions hold: $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ where $V_1 := \{(\dot{b}x, 0) : \dot{b} \in T_b^r \Gamma\}$, $V_2 := \{(xs(\dot{a}), \dot{a}y) : \dot{a} \in T_a^r \Gamma\}$, $W_1 := \{(\dot{b}x, \dot{b}z)\}$, $W_2 := \{(xs(\dot{a}), 0)\}$. Moreover $t_{id}(\dot{b}x, 0) = (\dot{b}x, \dot{b}z)$ and $t_{id}(xs(\dot{a}), \dot{a}y) = (xs(\dot{a}), 0)$. Using the definition of t_{id} and ω_0 we get the desired result. So the explicit formula for the product is: $\omega_1 \omega_2 = (f_1 * f_2) \omega_0$ and

$$(f_1 * f_2)(x) := \int_{F_l(x)} \lambda_0^2(y) f_1(y) f_2(s(y)x) = \int_{F_r(x)} \rho_0^2(y) f_1(xs(y)) f_2(y).$$

The second equality follows from the fact that right and left fibers are diffeomorphic by s . It is clear then it is smooth bidensity with support contained in $m(supp \omega_1, supp \omega_2)$ and this is compact set.

Corollary 4.6 We define the set $LM\mathcal{A}(\Gamma)$ of *left algebraic multipliers of $\mathcal{A}(\Gamma)$* as those linear mappings from $L\mathcal{A}(\Gamma)$ which commutes with right multiplication. This is evidently algebra. Then from the above proposition follows that \hat{h} is a homomorphism from $\mathcal{A}(\Gamma) \rightarrow LM\mathcal{A}(\Gamma')$.

That the multiplication is compatible with the star operation (i.e \mathcal{A} is in fact $*$ -algebra) can be shown directly, but again it will follow from more general facts which we prove later on.

The above defined multiplication is non degenerate as is easy seen from the following:

Lemma 4.7 For any $\omega \in \mathcal{A}(\Gamma)$: $\omega^* \omega = 0 \iff \omega = 0$.

Proof: Choose ω_0 and write $\omega = f \omega_0$. Then for $a \in \Gamma^0$ we have:

$$f^* * f(a) = \int_{\Gamma_a} \rho_0^2(x) f^*(s(x)) f(x) = \int_{\Gamma_a} \rho_0^2(x) \overline{f(x)} f(x).$$

From this formula the statement is clear.

■

It seems that there is no natural, geometric, norm on $\mathcal{A}(\Gamma)$ but one can introduce the family of useful norms “indexed” by ω_0 . [3] So choose λ_0 and write $\omega = f \omega_0$. We define quantities:

$$\|\omega\|_l := \sup_{a \in \Gamma^0} \int_{F_l(a)} \lambda_0^2 |f|, \quad \|\omega\|_r := \sup_{a \in \Gamma^0} \int_{F_r(a)} \rho_0^2 |f|, \quad \|\omega\| := \max\{\|\omega\|_l, \|\omega\|_r\}.$$

(We do not explicitly write the dependence from λ_0 to make notation simpler.) The next lemma shows that the definitions are correct.

Lemma 4.8 The functions $\|\cdot\|_l, \|\cdot\|_r, \|\cdot\|$ are norms and give \mathcal{A} the structure of normed algebra. Moreover $\|\omega^*\|_l = \|\omega\|_r$ so $\|\omega^*\| = \|\omega\|$. (in fact $(\mathcal{A}, *, \|\cdot\|)$ is normed $*$ -algebra as we will see later on.)

Proof: It is obvious that $\|\cdot\|_l, \|\cdot\|_r$ and consequently $\|\cdot\|$ are norms. Also from lemma 4.1 we immediately have $\|\omega\|_l = \|\omega^*\|_r$.

Let us show that $\|\omega_1 \omega_2\|_l \leq \|\omega_1\|_l \|\omega_2\|_l$. As usual we write: $\omega_1 = f_1 \omega_0$, $\omega_2 = f_2 \omega_0$. Then $\|\omega_1 \omega_2\|_l = \sup_{a \in \Gamma^0} \int_{a\Gamma} \lambda_0^2(x) |f_1 * f_2|(x)$.

$$\int_{a\Gamma} \lambda_0^2(x) |f_1 * f_2|(x) = \int_{a\Gamma} \lambda_0^2(x) \left| \int_{a\Gamma} \lambda_0^2(y) f_1(y) f_2(s(y)x) \right| \leq \int_{a\Gamma} \lambda_0^2(x) \int_{a\Gamma} \lambda_0^2(y) |f_1(y)| |f_2(s(y)x)| =$$

$$\begin{aligned}
&= \int_{a\Gamma \times a\Gamma} \lambda_0^2(x) \otimes \lambda_0^2(y) |f_1(y)| |f_2(s(y)x)| = \int_{a\Gamma \times \Gamma_a} \lambda_0^2(x) \otimes \rho_0^2(y) |f_1(s(y))| |f_2(yx)| = \\
&= \int_{\Gamma_a} \rho_0^2(y) |f_1(s(y))| \int_{a\Gamma} \lambda_0^2(x) |f_2(yx)| = \int_{\Gamma_a} \rho_0^2(y) |f_1(s(y))| \int_{F_1(y)} \lambda_0^2(z) |f_2(z)| \leq \\
&\leq \|\omega_2\|_l \int_{\Gamma_a} \rho_0^2(y) |f_1(s(y))| = \|\omega_2\|_l \int_{a\Gamma} \lambda_0^2(y) |f_2(y)| \leq \|\omega_2\|_l \|\omega_1\|_l.
\end{aligned}$$

So also $\sup_{a \in \Gamma^0} \int_{a\Gamma} \lambda_0^2(x) |f_1 * f_2|(x) \leq \|\omega_2\|_l \|\omega_1\|_l$. In the same way one can prove the inequality for $\|\cdot\|_r$ or use the fact that $(\mathcal{A}, *)$ is a *-algebra which is proved in lemma 4.10.

■

Remark 4.9 We can try to define a “geometric” norm on $\mathcal{A}(\Gamma)$ as follows. Recall that the orbit of a point $a \in E$ is a set: $e_L(\Gamma_a) = e_R(a\Gamma)$. It is known [5] that for each $a \in \Gamma^0$ the set $\Gamma_a \cap a\Gamma$ is a submanifold in Γ and a Lie group. Since right and left translations are diffeomorphisms of the fibers it is clear that all sets $a\Gamma \cap \Gamma_b$ for a, b in the same orbit are diffeomorphic submanifolds. Also we have that $e_L|_{\Gamma_a} : \Gamma_a \rightarrow \Gamma^0$ and $e_R|_{a\Gamma} : a\Gamma \rightarrow \Gamma^0$ are of constant rank so orbits are immersed submanifolds. *Now suppose that each orbit in Γ^0 is a submanifold* let us denote the orbit through a by O_a . In this case $(\Gamma_a, O_a, e_L|_{\Gamma_a})$ and $(a\Gamma, O_a, e_R|_{a\Gamma})$ are locally trivial differential fibrations, with the fibers diffeomorphic to a Lie group $a\Gamma \cap \Gamma_a$. If λ is a half density on Γ along the left fibers then its restriction to $a\Gamma$ can be written as $\lambda(x) = \mu(e_R(x)) \otimes \nu(x)$ for μ half density on O_a and ν half density on $a\Gamma$ along the fibers of $e_R|_{a\Gamma}$. In the same if ρ is a half density along the right fibers than its restriction to Γ_a can be written as $\rho(x) = \mu_1(e_L(x)) \otimes \nu_1(x)$ for μ_1 - half density on the orbit and ν_1 - half density along the fibers of $e_L|_{\Gamma_a}$. So for $\omega = \lambda \otimes \rho \in \mathcal{A}(\Gamma)$ and $x \in \Gamma$ with $e_L(x) = a, e_R(x) = b$ we have $\omega(x) = \mu(e_R(x)) \otimes \mu_1(e_L(x)) \otimes \nu(x) \otimes \nu_1(x)$ but since fibers of $e_R|_{\Gamma_a}$ and $e_L|_{b\Gamma}$ are the same $\nu(x) \otimes \nu_1(x)$ is a density on $T_x(\Gamma_a \cap b\Gamma)$. Let S denote the set of orbits and define the following quantity:

$$\|\omega\|_{geom} := \sup_{s \in S} \sqrt{\int_{s \times s} |\mu_1|^2(a) \otimes |\mu|^2(b) \left(\int_{a\Gamma \cap \Gamma_b} |\nu \nu_1| \right)^2}.$$

Let us show that this quantity is finite and for $\omega \in \mathcal{A}(\Gamma)$: $\|\omega\|_{geom} \leq \|\omega\|$ where the norm on the right side is introduced above.

Let $\omega = f \lambda_0 \otimes \rho_0$, $\lambda_0(x) =: \mu(b) \otimes \nu(x)$, $\rho_0(x) =: \mu_1(a) \otimes \nu_1(x)$, $a := e_L(x)$, $b = e_R(x)$. Choose an orbit O_c . We have to estimate integral:

$$I(c) := \int_{O_c \times O_c} (\mu_1)^2(a) \otimes (\mu)^2(b) \left[\int_{a\Gamma \cap \Gamma_b} |f| |\nu| |\nu_1| \right]^2.$$

First we use the Schwarz inequality for integral over $a\Gamma \cap \Gamma_b$:

$$\int_{a\Gamma \cap \Gamma_b} |f| |\nu| |\nu_1| \leq \left(\int_{a\Gamma \cap \Gamma_b} |f| \nu^2 \right)^{1/2} \left(\int_{a\Gamma \cap \Gamma_b} |f| \nu_1^2 \right)^{1/2}.$$

In this way we get:

$$I(c) \leq \int_{O_c \times O_c} (\mu_1)^2(a) \otimes \mu^2(b) g(a, b) g_1(a, b),$$

where: $g(a, b) := \int_{a\Gamma \cap \Gamma_b} |f| \nu^2$ and $g_1(a, b) := \int_{a\Gamma \cap \Gamma_b} |f| \nu_1^2$. Then g, g_1 are compactly supported, smooth functions on $O_c \times O_c$. Let $g_1(a, b_0) := \sup_{b \in O_c} \{g_1(a, b)\}$. Now:

$$\begin{aligned}
\int_{O_c \times O_c} (\mu_1)^2(a) \otimes \mu^2(b) g(a, b) g_1(a, b) &= \int_{O_c} (\mu_1)^2(a) \int_{O_c} \mu^2(b) g_1(a, b) g(a, b) \leq \\
&\int_{O_c} (\mu_1)^2(a) g_1(a, b_0) \int_{O_c} \mu^2(b) g(a, b).
\end{aligned}$$

But

$$\int_{O_c} \mu^2(b) g(a, b) = \int_{O_c} \mu^2(b) \int_{a\Gamma \cap \Gamma_b} |f| \nu^2 = \int_{a\Gamma} |f| \lambda_0^2 \leq \|\omega\|_l.$$

In this way:

$$I(c) \leq \|\omega\|_l \int_{O_c} (\mu_1)^2(a) g_1(a, b_0) \leq \|\omega\|_l \int_{O_c} (\mu_1)^2(a) \int_{\Gamma \cap \Gamma_{b_0}} |f| \nu_1^2 = \|\omega\|_l \int \Gamma_{b_0} |f| \rho_0^2 \leq \|\omega\|_l \|\omega\|_r.$$

So for each orbit $s \in S$ we have $\sqrt{I(s)} \leq \sqrt{\|\omega\|_l \|\omega\|_r} \leq \|\omega\|$ and finally: $\|\omega\|_{geom} \leq \|\omega\|$.

Now we show that \mathcal{A} is a normed *-algebra and morphisms define *-homomorphisms. As in the Prop.4.3 the equality we prove is taken from the definition of a conjugation of a linear mapping on C^* -algebra.

Proposition 4.10 *Let $h : \Gamma \rightarrow \Gamma'$ be a morphism of differential groupoids. Then*

$$\omega_3^*(\hat{h}(\omega_1)(\omega_2)) = (\hat{h}(\omega_1^*)\omega_3)^*\omega_2 \text{ for any } \omega_1 \in \mathcal{A}(\Gamma), \omega_2, \omega_3 \in \mathcal{A}(\Gamma').$$

Proof: Choose ω_0 and ω'_0 - *-invariant bidensities on Γ and Γ' and write: $\omega_1 = f_1 \omega_0$, $\omega_2 = f_2 \omega'_0$, $\omega_3 = f_3 \omega'_0$. Then $\omega_3^*(\hat{h}(\omega_1)(\omega_2))(z) = (f_3^* * (f_1 *_h f_2))(z) \omega'_0(z)$, where

$$(f_3^* * (f_1 *_h f_2))(z) = \int_{F_r(z)} \rho_0'^2(y) f_3^*(zs(y)) (f_1 *_h f_2)(y) = \int_{\Gamma'_a} \rho_0'^2(y) f_3^*(s(y)) (f_1 *_h f_2)(yz), \quad a := e'_L(z)$$

in this equality we have used diffeomorphism $F_r(z) \simeq \Gamma'_a$. Using the definition of \hat{h} we can write this expression as:

$$\int_{\Gamma'_a} \rho_0'^2(y) \overline{f_3(y)} \int_{F_l(f_h(b))} \lambda_0^2(x) f_1(x) t_h(x, y_1) f_2(y_1),$$

where: $b := e'_L(y)$, $y_1 := s'(h_b^L(x))yz$.

This integral can be interpreted as integral over $\Gamma *_h \Gamma'_a$ of the density:

$$\Psi_z(x, y) := (\lambda_0^2(x) \otimes \rho_0'^2(y)) (\overline{f_3(y)} f_1(x) t_h(x, y_1) f_2(y_1)).$$

The right hand side: $((\hat{h}(\omega_1^*)\omega_3)^*\omega_2)(z) = ((f_1^* *_h f_3)^* * f_2)(z) \omega'_0(z)$.

$$\begin{aligned} ((f_1^* *_h f_3)^* * f_2)(z) &= \int_{F_l(a)} \lambda_0'^2(y_2) (f_1^* *_h f_3)^*(y_2) f_2(s(y_2)z) = \int_{F_l(a)} \lambda_0'^2(y_2) \overline{f_1^* *_h f_3(s(y_2))} f_2(s(y_2)z) = \\ &= \int_{F_l(a)} \lambda_0'^2(y_2) \int_{F_l(f_h(b_2))} \lambda_0^2(x_2) f_1(s(x_2)) t_h(x_2, y_3) \overline{f_3(y_3)} f_2(s(y_2)z), \end{aligned}$$

where: $b_2 := e'_R(y_2)$, $y_3 := s'(h_{b_2}^L(x_2))s'(y_2)$.

Now we interpret the integral as a integral over the submanifold

$$\Gamma_h \times F_l(a) := \{(x_2, y_2) \in \Gamma \times \Gamma' : e'_L(y_2) = a, e_L(x_2) = f_h(e'_L(y_2))\}$$

of the density:

$$\Phi_z(x_2, y_2) := (\lambda_0^2(x_2) \otimes \lambda_0'^2(y_2)) f_1(s(x_2)) t_h(x_2, y_3) \overline{f_3(y_3)} f_2(s(y_2)z).$$

The mapping $s \times s'$ is a diffeomorphism from $\Gamma_h \times F_l(a)$ onto $\Gamma \times_h \Gamma'_a$, moreover

$$(s \times s')(\lambda_0^2(x_2) \otimes \lambda_0'^2(y_2)) = \rho_0^2(s(x_2)) \otimes \rho_0'^2(s'(y_2)).$$

From this it follows that the above integral is equal to the integral over $\Gamma \times_h \Gamma'_a$ of a density:

$$\tilde{\Phi}_z(x_3, y_4) := (\rho_0^2(x_3) \otimes \rho_0'^2(y_4)) f_1(x_3) t_h(s(x_3), y_3) \overline{f_3(y_3)} f_2(y_4 z),$$

where $y_3 := h_{b_2}^R(x_3)y_4$, $b_2 := e'_L(y_4)$.

Now we use $t_h : \Gamma \times_h \Gamma'_a \rightarrow \Gamma *_h \Gamma'_a$ and get:

$$(t_h \tilde{\Phi}_z)(x, y) := (\lambda_0^2(x) \otimes \rho_0'^2(y)) f_1(x) \overline{f_3(y)} f_2(y_5 z) t_h^2(x, y_5) t_h(s(x), y),$$

where $(x, y) \in \Gamma *_h \Gamma'_a$, $y_5 := s'(h_b^L(x))y$, $b := e'_L(y)$.

This is equal to $\Psi_z(x, y)$ provided that $t_h(x, y_5 z) = t_h^2(x, y_5) t_h(s(x), y)$. From the lemma 4.4: $t_h(x, y_5) = t_h(x, y_5 z)$, so it remains to show equality: $1 = t_h(x, y_5) t_h(s(x), y)$. This is done in the following:

Lemma 4.11 Let $(x, y) \in \Gamma *_h \Gamma'_a$ and $b := e'_L(y)$. Then $t_h(s(x), y)t_h(x, s'(h_b^L(x)y)) = 1$.

Proof: It is easy to check that $t_h(s \times id)t_h = (s \times id)$ and $(s \times id)i_1(\rho_0(x) \otimes \rho'_0(y)) = i_2(\lambda_0(s(x)) \otimes \rho'_0(y))$. We compute:

$$\begin{aligned} t_h(s \times id)t_h i_1(\rho_0(x) \otimes \rho'_0(y)) &= t_h(x, y)t_h(s \times id)i_2(\lambda_0(x) \otimes \rho'_0(y')) = \\ &= t_h(x, y)t_h i_1(\rho_0(s(x)) \otimes \rho'_0(y')) = t_h(x, y)t_h(s(x), y')i_2(\lambda_0(s(x)) \otimes \rho'_0(y)). \end{aligned}$$

So $t_h(x, y)t_h(s(x), y') = 1$.

■

Corollary 4.12 **-algebra structure on \mathcal{A} .* Take $h = id$ then we have $\omega_3^*(\omega_1\omega_2) = (\omega_3^*\omega_1)\omega_2 = (\omega_1^*\omega_3)^*\omega_2$ for any bidensities $\omega_1, \omega_2, \omega_3$. So $(\omega_3^*\omega_1) = (\omega_1^*\omega_3)^*$ (multiplication is non degenerate), since $*$ is an involution we have that $(\omega_1\omega_2)^* = \omega_2^*\omega_1^*$. This together with lemma 4.8 shows that $(\mathcal{A}, ||.||)$ is a normed *-algebra.

Representation of *-algebra of groupoid associated with a morphism.

Now we are going to show that any morphism $h : \Gamma \rightarrow \Gamma'$ of differential groupoids defines representation of *-algebra $\mathcal{A}(\Gamma)$ in the Hilbert space $L^2(\Gamma')$ of square integrable half densities on Γ' . Again we use lemma 3.5. Let Ψ be a smooth half density on Γ' with compact support and $\omega \in \mathcal{A}(\Gamma)$, $\omega = \lambda \otimes \rho$. Let $(x, y) \in \Gamma \times_h \Gamma'$ and $t_h(x, y) = (x, z)$. As in the definition of \hat{h} , $\rho(x) \otimes \Psi(y)$ can be viewed as a half density on $T_{(x,y)}(\Gamma \times_h \Gamma')$ and $t_h(\rho(x) \otimes \Psi(y))$ is a half density on $T_{(x,z)}(\Gamma * \Gamma')$. Since $\Omega_L^{1/2}(x) \otimes \Omega^{1/2}T_z\Gamma' \simeq \Omega^{1/2}T_{(x,z)}(\Gamma *_h \Gamma')$ this half density can be written as $\tilde{\lambda}_x \otimes \Psi_x(z)$ for $\tilde{\lambda}_x$ —a half density on $T_x(F_l(x))$ and $\Psi_x(z)$ —a half density on $T_z(\Gamma')$. Then $\lambda(x)\tilde{\lambda}_x \otimes \Psi_x(z)$ is a 1-density on $T_x^*\Gamma$ with values in half densities on $T_z(\Gamma')$. Integrating it we get half density on $T_z(\Gamma')$. Let us define:

$$(\pi_h(\omega)\Psi)(z) := \int_{F_l(f(a))} [\lambda(x)\tilde{\lambda}(x)] \otimes \Psi_x(z).$$

Choose ω_0 and write $\omega = f\omega_0$. Since e'_R is a surjective submersion we have: $\Omega^{1/2}T_w\Gamma' \simeq \Omega_R^{1/2}(w) \otimes \Omega^{1/2}T_{e'_R(w)}E'$ for any $w \in \Gamma'$. In this way, if we choose ρ'_0 and ν_0 —non vanishing, real half density on E' then $\rho'_0 \otimes \nu_0$ defines non -vanishing, real, half density on Γ' . So any other smooth half density with compact support Ψ can be written as $\Psi = \psi \rho'_0 \otimes \nu_0 =: \psi \Psi_0$ for some smooth, complex function ψ with compact support. It is easy to see that: $t_h(\rho_0(x) \otimes \rho'_0(y) \otimes \nu_0(a)) = t_h(x, y)\lambda_0(x) \otimes \rho'_0(z) \otimes \nu_0(a)$ where t_h is as in the definition of \hat{h} . So the explicit formula is:

$$(\pi_h(\omega)\Psi)(z) = \left[\int_{b\Gamma} \lambda_0^2(x) f(x) t_h(x, y) \psi(y) \right] \Psi_0(z),$$

where $b := f_h(e'_L(z))$, $t_h(x, y) = (x, z)$. Note that formally the expression is the same as in Prop. 4.3.

Proposition 4.13 a) Let $h : G \rightarrow \Gamma'$ be a morphism of differential groupoids. Let $||.||$ be a norm on $\mathcal{A}(\Gamma)$ associated with choosen ω_0 . The correspondance: $\mathcal{A}(\Gamma) \ni \omega \mapsto \pi_h(\omega)$ is representation of the normed *-algebra $\mathcal{A}(\Gamma)$ in $L^2(\Gamma')$.

b) If $k : \Gamma' \rightarrow \Gamma''$ is a morphism then: $\pi_k(\hat{h}(\omega_1)\omega_2)\Psi = \pi_{kh}(\omega_1)\pi_k(\omega_2)\Psi$ for any $\omega_1 \in \mathcal{A}(\Gamma)$, $\omega_2 \in \mathcal{A}(\Gamma')$, Ψ —smooth half density on Γ'' with compact support.

Proof: Let $\Psi = \psi \Psi_0$ for $\Psi_0 := \rho'_0 \otimes \nu_0$ and let $\omega = f\omega_0$.

b) This follows directly from Prop. 4.3.

a) Let $(,)$ be a scalar product in $L^2(\Gamma')$.

$$|(\Psi, \pi_h(\omega)\Psi)| = \left| \int_{\Gamma'} \overline{\psi(z)} \Psi_0^2(z) \int_{b\Gamma} \lambda_0^2(x) f(x) t_h(x, y) \psi(y) \right|,$$

where $b := f_h(e'_L(z))$ and $t_h(x, y) = (x, z)$.

Using the definition of Ψ_0 one can estimate:

$$\left| \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(z) \overline{\psi(z)} \int_{b\Gamma} \lambda_0^2(x) f(x) t_h(x, y) \psi(y) \right| \leq \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(z) |\psi(z)| \int_{b\Gamma} \lambda_0^2(x) |f(x) t_h(x, y) \psi(y)|.$$

For fixed $a \in E'$ the integral

$$\int_{\Gamma'_a} \rho_0'^2(z) |\psi(z)| \int_{\Gamma} \lambda_0^2(x) |f(x)| |t_h(x, y)| |\psi(y)|$$

can be viewed as the integral over $\Gamma *_h \Gamma'_a$ of a 1-density

$$(\lambda_0^2(x) \otimes \rho_0'^2(z)) |\psi(z) f(x) t_h(x, y) \psi(y)|$$

and it is equal to the scalar product in $L^2(\Gamma *_h \Gamma'_a)$ of half densities (φ_1, φ_2) for

$$\varphi_1(x, z) := \sqrt{|f(x)|} |\psi(z)| \operatorname{sgn}(t_h(x, y)) \lambda_0(x) \otimes \rho_0'(z),$$

$$\varphi_2(x, z) := \sqrt{|f(x)|} t(x, y) |\psi(y)|, (\lambda_0(x) \otimes \rho_0'(z)).$$

(since $t_h(x, y) \neq 0$, $\operatorname{sgn}(t_h)$ is well defined, smooth function). But since $\mathbf{t}_h : \Gamma \times_h \Gamma'_a \longrightarrow \Gamma *_h \Gamma'_a$ is a diffeomorphism, it defines unitary operator $\mathbf{t}_h : L^2(\Gamma \times_h \Gamma'_a) \longrightarrow L^2(\Gamma *_h \Gamma'_a)$ and $\varphi_2 = \mathbf{t}_h \tilde{\varphi}_2$ where $\tilde{\varphi}_2(x, y) := \sqrt{|f(x)|} |\psi(y)| (\rho_0(x) \otimes \rho_0'(y))$ —is a half density on $\Gamma' \times_h \Gamma'_a$. So we have: $|(\varphi_1, \varphi_2)| \leq \|\varphi_1\| \|\mathbf{t}_h \tilde{\varphi}_2\| = \|\varphi_1\| \|\tilde{\varphi}_2\|$.

$$\begin{aligned} \|\varphi_1\|^2 &= \int_{\Gamma *_h \Gamma'_a} |\varphi_1|^2 = \int_{\Gamma'_a} \rho_0'^2(z) |\psi(z)|^2 \int_{\Gamma} \lambda_0^2(x) |f(x)| \leq \|\omega\|_L \int_{\Gamma'_a} \rho_0'^2(z) |\psi(z)|^2. \\ \|\tilde{\varphi}_2\|^2 &= \int_{\Gamma \times_h \Gamma'_a} |\tilde{\varphi}_2|^2 = \int_{\Gamma'_a} \rho_0'^2(y) |\psi(y)|^2 \int_{\Gamma_c} \rho_0^2(x) |f(x)| \leq \|\omega\|_R \int_{\Gamma'_a} \rho_0'^2(y) |\psi(y)|^2, \end{aligned}$$

where $c := f_h(e'_L(y))$.

So finally we get an estimate:

$$|(\Psi, \pi_h(\omega)\Psi)| \leq \int_{E'} \nu_0^2(a) \sqrt{\|\omega\|_L \|\omega\|_R} \int_{\Gamma'_a} \rho_0'^2(z) |\psi(z)|^2 = \sqrt{\|\omega\|_L \|\omega\|_R} \|\Psi\|^2 \leq \|\omega\| \|\Psi\|^2.$$

This shows that the operator $\pi_h(\omega)$ (defined on smooth half densities with compact support) is bounded. Since smooth half densities with compact support are dense in $L^2(\Gamma')$, $\pi_h(\omega)$ can be uniquely extended to a bounded operator on $L^2(\Gamma')$.

Putting $h = id : \Gamma \rightrightarrows \Gamma$ we get from b): $\pi_k(\omega_1 \omega_2)\Psi = \pi_k(\omega_1)\pi_k(\omega_2)\Psi$. This shows that π_k is a representation.

Now we show that $\pi_h(\omega^*) = (\pi_h(\omega))^*$.

$$\begin{aligned} (\pi_h(\omega^*)\Psi, \Psi) &= \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(z) \psi(z) \overline{\int_{\Gamma} \lambda_0^2(x) f^*(x) t_h(x, y) \psi(y)} = \\ &= \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(z) \psi(z) \int_{\Gamma} \lambda_0^2(x) f(s(x)) t_h(x, y) \overline{\psi(y)}, \end{aligned}$$

where $c := f_h(e'_L(z))$, $\mathbf{t}_h(x, y) = (x, z)$.

In the same way:

$$(\Psi, \pi_h(\omega)\Psi) = \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(z) \overline{\psi(z)} \int_{\Gamma} \lambda_0^2(x) f(x) t_h(x, y) \psi(y).$$

We will show that:

$$\int_{\Gamma'_a} \rho_0'^2(z) \psi(z) \int_{\Gamma} \lambda_0^2(x) f(s(x)) t_h(x, y) \overline{\psi(y)} = \int_{\Gamma'_a} \rho_0'^2(z) \overline{\psi(z)} \int_{\Gamma} \lambda_0^2(x) f(x) t_h(x, y) \psi(y).$$

In the Prop. 4.10 the following equality was proved:

$$\int_{\Gamma'_a} \rho_0'^2(y) \overline{f_3(y)} \int_{\Gamma} \lambda_0^2(x) f_1(x) t_h(x, y_1) f_2(y_1) = \int_{\Gamma'_a} \rho_0'^2(y_2) \int_{\Gamma} \lambda_0^2(x_2) f_1(s(x_2)) t_h(x_2, y_3) \overline{f_3(y_3)} f_2(s(y_2)),$$

where $c := f_h(e'_L(y))$, $t_h(x, y_1) = (x, y)$, $c_2 := f_h(e'_R(y_2))$, $t_h(x_2, y_3) = (x_2, s(y_2))$, f_2, f_3 are smooth function on Γ' with compact support and f_1 is smooth function on Γ with compact support.

Using s' we can rewrite the right hand side of the equality as:

$$\int_{\Gamma'_a} \rho_0'^2(y_4) \int_{c_4\Gamma} \lambda_0^2(x_2) f_1(s(x_2)) t_h(x_2, y_3) \overline{f_3(y_3)} f_2(y_4),$$

where $c_4 := f_h(e'_L(y_4))$, $t_h(x_2, y_3) = (x_2, y_4)$.

Now put $f_2 = f_3 = \psi$, $f_1 = f$. We get:

$$\begin{aligned} & \int_{\Gamma'_a} \rho_0'^2(y) \overline{\psi(y)} \int_{c\Gamma} \lambda_0^2(x) f(x) t_h(x, y_1) \psi(y_1) = \\ & = \int_{\Gamma'_a} \rho_0'^2(y) \psi(y) \int_{c\Gamma} \lambda_0^2(x) f(s(x)) t_h(x, y_1) \overline{\psi(y_1)}. \end{aligned}$$

And this is desired equality.

■

Examples 4.14 *a) Reduced C^* -algebra of a differential groupoid.* Let l be the morphism from Γ to the pair groupoid $\Gamma \times \Gamma$ defined in Example 2.11 f) i.e. $(x, y; z) \in Gr(l) \iff (x; z, y) \in Gr(m)$. It is easy to see that in this case: $f_l = e_L$, $\Gamma \times_l (\Gamma \times \Gamma) = \{(x, y, z) \in \Gamma \times \Gamma \times \Gamma : e_R(x) = e_L(y)\}$, $\Gamma *_l (\Gamma \times \Gamma) = \{(x, y, z) \in \Gamma \times \Gamma \times \Gamma : e_L(x) = e_L(y)\}$ and $t_l(x; y, z) = (x; xy, z)$. π_l is a representation of $\mathcal{A}(\Gamma)$ in $L^2(\Gamma \times \Gamma) = L^2(\Gamma) \otimes L^2(\Gamma)$ and a short computation shows that $\pi_l = \pi_{id} \otimes I$. So $\|\pi_l(\omega)\| = \|\pi_{id}(\omega)\|$. Also from Lemma 4.7 easily follows that $\mathcal{A}(\Gamma) \ni \omega \mapsto \|\pi_{id}(\omega)\|$ is a C^* norm on $\mathcal{A}(\Gamma)$. The completion of $\mathcal{A}(\Gamma)$ in this norm will be called *reduced C^* -algebra of Γ* and denoted by $C_{red}^*(\Gamma)$.

b) Modular function. Let \tilde{e} be a morphism from Γ to the pair groupoid $E \times E$ defined in Example 2.11 e), i.e. $Gr(\tilde{e}) = \{(e_L(x), e_R(x); x) : x \in \Gamma\}$. It is easy to see that: $\Gamma \times_{\tilde{e}} (E \times E) = \{(x; e_R(x), e) : x \in \Gamma, e \in E\}$, $\Gamma *_{\tilde{e}} (E \times E) = \{(x; e_L(x), e) : x \in \Gamma, e \in E\}$, $m_{\tilde{e}}(x; e_R(x), e) = (e_L(x), e)$ and $t_{\tilde{e}}(x; e_R(x), e) = (x; e_L(x), e)$. Choose some $\omega_0 = \lambda_0 \otimes \rho_0$ and some real, non vanishing half density ν_0 on E . Such choice defines function $t_{\tilde{e}}(x; e_R(x), e)$. From Lemma 4.4, this function does not depend from e and if we define $\Delta(x) := t_{\tilde{e}}(x, e_R(x))$ then $\Delta(xy) = \Delta(x)\Delta(y)$ for any composable $x, y \in \Gamma$. Δ is called *modular function of Γ* (it depends from chosen λ_0, ν_0). The function Δ can also be described in the following way. When ω_0, ν_0 are choosen, the expressions: $\psi_r(x) := \rho_0(x) \otimes \nu_0(e_R(x))$ and $\psi_l(x) := \lambda_0(x) \otimes \nu_0(e_L(x))$ define smooth, non vanishing, real half densities on Γ . Then Δ is defined by: $\psi_l =: \Delta\psi_r$.

Remark 4.15 Dependence of Δ on choice of λ_0 and ν_0 can be described in the cohomological way. Let us define $\Gamma^{(0)} := \Gamma^0$, $\Gamma^{(1)} := \Gamma$ and, for $n \geq 2$, $\Gamma^{(n)} := \{(x_0, \dots, x_{n-1}) \in \Gamma \times \dots \times \Gamma : e_R(x_i) = e_L(x_{i+1}), i = 0, \dots, n-1\}$.

(Smooth) n -cochain is a smooth function $f : \Gamma^{(n)} \longrightarrow \mathbb{R} \setminus \{0\}$ which, for $n > 0$, satisfies condition:

$$[\exists i \in \{0, \dots, n-1\} : x_i \in \Gamma^0] \Rightarrow f(x_0, \dots, x_i, \dots, x_{n-1}) = 1.$$

Group of n -cochains (with a pointwise multiplication) we denote by $C^n(\Gamma)$.

Define coboundary operators $\delta^n : C^n(\Gamma) \longrightarrow C^{n+1}(\Gamma)$:

$$\begin{aligned} (\delta^0 f)(x) &:= \frac{f(e_L(x))}{f(e_R(x))} \text{ and, for } n > 0, \\ (\delta^n f)(x_0, x_1, \dots, x_n) &:= \\ &= f(x_1, \dots, x_n) \prod_{i=1}^n (f(x_0, \dots, x_{i-1}x_i, \dots, x_n))^{s(i)} (f(x_0, \dots, x_{n-1}))^{s(n+1)}, \end{aligned}$$

where $s(i) := (-1)^i$.

It is easy to check that $\delta^{n+1}\delta^n = 1$. In this way we get complex and cocycles, coboundaries and cohomology groups [3].

Now, let $\tilde{\lambda}_0, \tilde{\nu}_0$ be other half densities. We have $\tilde{\lambda}_0(x) = f(e_R(x))\lambda_0(x)$ and $\tilde{\nu}_0(a) = g(a)\nu_0(a)$ for some smooth, non vanishing, real functions on Γ^0 . Then from the equality $\tilde{\psi}_l = \tilde{\Delta}\tilde{\psi}_r$ we get

$$\tilde{\Delta}(x) = \Delta(x) \frac{f(e_R(x))g(e_L(x))}{f(e_L(x))g(e_R(x))} = \Delta(x)(\delta^0 \frac{g}{f})(x).$$

So Δ i $\tilde{\Delta}$ are in the same cohomology class.

Nondegeneracy of morphisms.

Now we are going to show that the action of morphisms on bidensities is nondegenerate (i.e. if $\hat{h}(\omega)\omega' = 0$ for any ω then $\omega' = 0$.) This is important in the context of morphisms of C^* -algebras. In fact we prove more general:

Proposition 4.16 *Let $h : \Gamma \rightarrow \Gamma'$ be a morphism of differential groupoids. Then for any $\omega' \in \mathcal{A}(\Gamma')$ there exists a sequence $\omega_n \in \mathcal{A}(\Gamma)$ such that: $\lim_{n \rightarrow \infty} \hat{h}(\omega_n)\omega' = \omega'$. (The limit is in topology defined by some $\|\cdot\|$ of above defined type on Γ' .)*

The proof is based on several lemmas and is slightly modified version of the proof given in [3]:

Lemma 4.17 *Let $h : \Gamma \rightarrow \Gamma'$ be a morphism of differential groupoids and let $K \subset \Gamma'$ be a compact subset. Then exists U_K - open neighbourhood of Γ^0 in Γ such that $\overline{m_h((U_K \times K) \cap (\Gamma \times_h \Gamma'))}$ is compact.*

Proof:

1. $m_h : \Gamma \times_h \Gamma' \rightarrow \Gamma'$ is a smooth mapping and $m_h(a, z) = z$ for $z \in \Gamma'$, $a := f_h(e'_L(z))$, so for any neighbourhood $O_z \ni z$ there exist neighbourhoods $\Gamma \supset V_a \ni a$ and $O'_z \ni z$ such that $m_h((V_a \times O'_z) \cap (\Gamma \times_h \Gamma')) \subset O_z$. Moreover we can assume that $e_R^{-1}(V_a \cap \Gamma^0) \cap V_a = V_a$, $O'_z \subset O_z$, $f_h(e'_L(O'_z)) \subset V_a$.
2. Let $\{O_z, z \in K\}$, $z \in O_z$ be an open covering of K such that $\overline{O_z}$ is compact. From the previous point the family $\{O'_z, z \in K\}$ is an open covering of K and $\{V_a, a = f_h(e'_L(z)), z \in K\}$ an open covering of a compact set $H := f_h(e'_L(K))$. So one can choose finite covering $O'_{z_1} \cup \dots \cup O'_{z_m} \supset K$ with the corresponding V_{a_1}, \dots, V_{a_m} and O_{z_1}, \dots, O_{z_m} . Then $H \subset V_1 \cup \dots \cup V_m$, where $V_i := V_{a_i}$.
3. For $x \in H$ let W_x be an open (in Γ^0) neighbourhood of x contained in $V_1 \cup \dots \cup V_m$. Define $U_x := e_R^{-1}(W_x) \cap V_{i_1} \cap \dots \cap V_{i_l}$ where the intersection is with these sets V_i which contain x . The family $\{U_x, x \in H\}$ is an open (in Γ) covering of H so we can choose $U_1 \cup \dots \cup U_n =: U \supset H$.
4. Now let $(x, z) \in (U \times K) \cap (\Gamma \times_h \Gamma')$. z is contained in some O'_{z_i} and x in some U_j . Since $e_R(x) = f_h(e'_L(z))$, $e_R(x) \in V_i$ and from the construction also $x \in V_i$ and then $m_h(x, z) \in O_{z_i}$. So $m_h((U \times K) \cap (\Gamma \times_h \Gamma')) \subset O_{z_1} \cup \dots \cup O_{z_m} \subset \overline{O_{z_1}} \cup \dots \cup \overline{O_{z_m}}$ and this is compact set.
5. Finally one defines $U_K := U \cup \bigcup_{x \in \Gamma^0 \setminus H} e_R^{-1}(O_x)$ where O_x is an open (in Γ^0) neighbourhood of x and $O_x \cap H = \emptyset$.

■

Lemma 4.18 *Let $h : \Gamma \rightarrow \Gamma'$ be a morphism of differential groupoids. If g is a continous function on Γ' and $K \subset \Gamma'$ is compact then for any $\delta > 0$ there exists U^δ - open neighbourhood of Γ^0 in Γ such that:*

$$(x, z) \in (U^\delta \times K) \cap (\Gamma *_h \Gamma') \Rightarrow |g(s'(h_a^L(x))z) - g(z)| \leq \delta, \text{ where } a := e'_L(z).$$

Proof: Choose $\delta > 0$. Then for any $z \in K$ there exists O_z^δ - open neighborhood of z such that: $y \in O_z^\delta \Rightarrow |g(y) - g(z)| \leq \delta/2$. The family $\{O_z^\delta, z \in K\}$ is a covering of K . Put $U_1^\delta := U_K$ where U_K is constructed from $\{O_z^\delta\}$ as in the previous lemma. Then for $(x, z) \in (U_1^\delta \times K) \cap (\Gamma \times_h \Gamma')$ we have $z \in O_{z_i}^\delta$ for some $z_i \in K$ and $m_h(x, z) \in O_{z_i}^\delta$, then $|g(m_h(x, z)) - g(z)| = |g(m_h(x, z)) - g(z_i) + g(z_i) - g(z)| \leq \delta/2 + \delta/2 = \delta$. Finally we define $U^\delta := s(U_1^\delta)$ and the result follows from $(x, z) \in (\Gamma *_h \Gamma') \iff (s(x), z) \in \Gamma \times_h \Gamma'$ and $s'(h_a^L(x)) = h_a^R(s(x))$.

■

Lemma 4.19 *Let $h : \Gamma \rightarrow \Gamma'$ be a morphism of differential groupoids. Choose ω_0, ω'_0 – this defines function t_h . Let $K \subset \Gamma'$ be compact and let $1 > \delta > 0$. Then there exists $U_1^\delta \subset \Gamma$ – open neighbourhood of Γ^0 such that:*

$$(x, z) \in (U_1^\delta \times K) \cap (\Gamma *_h \Gamma') \Rightarrow |t_h(x, s'(h_a^L(x))z) - 1| \leq \delta, \text{ where } a := e'_L(z).$$

Proof: We begin by showing that $t_h(f_h(b), b) = 1$ for any $b \in E'$. Let $a := f_h(b)$, $V := T_{(a,b)}(\Gamma \times_h \Gamma'_b)$, $W := T_{(a,b)}(\Gamma *_h \Gamma'_b)$. Let (X_1, \dots, X_k) be a basis in $T_a^r \Gamma$ and (Y_1, \dots, Y_m) basis in $T_b^r \Gamma'$. Then $(s(X_1), \dots, s(X_k))$ is a basis in $T_a^l \Gamma$, $(\hat{X}_1, \dots, \hat{X}_k, \hat{Y}_1, \dots, \hat{Y}_m)$ is a basis in V and $(\tilde{X}_1, \dots, \tilde{X}_k, \hat{Y}_1, \dots, \hat{Y}_m)$ is a basis in W . Where $\hat{X}_i := (X_i, 0_b)$, $\tilde{X}_i := (s(X_i), 0_b)$, $\hat{Y}_i := (f_h e'_L(Y_i), Y_i)$.

The isomorphism $i_1 : \Omega_R^{1/2}(a) \otimes \Omega_R^{1/2}(b) \rightarrow \Omega^{1/2}V$ is given by the formula:

$$i_1(\rho_0(a) \otimes \rho'_0(b))(\hat{X}_1 \wedge \dots \wedge \hat{Y}_m) := \rho_0(a)(X_1 \wedge \dots \wedge X_k) \rho'_0(b)(Y_1 \wedge \dots \wedge Y_m)$$

and $i_2 : \Omega_L^{1/2}(a) \otimes \rho_0(b) \rightarrow \Omega^{1/2}W$ by:

$$i_2(\lambda_0(a) \otimes \rho'_0(b))(\tilde{X}_1 \wedge \dots \wedge \hat{Y}_m) := \lambda_0(a)(s(X_1) \wedge \dots \wedge s(X_k)) \rho'_0(b)(Y_1 \wedge \dots \wedge Y_m).$$

Moreover $t_h(\hat{Y}_i) = \hat{Y}_i$ and $t_h(\tilde{X}_i) = (X_i, h_b^R(X_i)b) =: \alpha_{il}\tilde{X}_l + \beta_{ij}\hat{Y}_j$. So

$$\begin{aligned} i_2(\lambda_0(a) \otimes \rho'_0(b))(t_h \tilde{X}_1 \wedge \dots \wedge t_h \hat{Y}_m) &= |\det \alpha|^{1/2} \lambda_0(a)(s(X_1) \wedge \dots \wedge s(X_k)) \rho'_0(b)(Y_1 \wedge \dots \wedge Y_m) = \\ &= |\det \alpha|^{1/2} \rho_0(a)(X_1 \wedge \dots \wedge X_k) \rho'_0(b)(Y_1 \wedge \dots \wedge Y_m) \end{aligned}$$

and

$$(t_h i_1)(\rho_0(a) \otimes \rho'_0(b))(t_h \hat{X}_1 \wedge \dots \wedge t_h \hat{Y}_m) = \rho_0(a)(X_1 \wedge \dots \wedge X_k) \rho'_0(b)(Y_1 \wedge \dots \wedge Y_m).$$

In this way $t_h(a, b) = |\det \alpha|^{1/2}$. From the definition of α :

$$(X_i, h_b^R(X_i)b) = \alpha_{il}\tilde{X}_l + \beta_{ij}\hat{Y}_j = \alpha_{il}(s(X_l), 0_b) + \beta_{ij}(f_h e'_L(Y_j), Y_j).$$

So $X_i = \alpha_{il}s(X_l) + \beta_{ij}f_h e'_L(Y_j)$ and this is decomposition of X_j with respect to the direct sum $T_a \Gamma = T_a^l \Gamma \oplus T_a E$. Applying s to this decomposition we easy get $\alpha^2 = 1$, so $|\det \alpha| = 1$.

Let $1 > \delta > 0$ be given and let $H \in E'$ be compact. Arguing as in the proof of Lemma 4.17 we can find $H \subset O_H$ – open in Γ' and U_H – an open neighbourhood of Γ^0 in Γ such that:

$$(x, y) \in (U_H \times O_H) \cap (\Gamma \times_h \Gamma') \Rightarrow |t_h(x, y) - 1| \leq \delta/2.$$

From Lemma 4.4 $t_h(x, yz) = t_h(x, y)$ so above estimate is valid for $y \in \Gamma'$ with $e'_L(y) \in H$. So putting $H := e'_L(K)$ we get:

$$(x, y) \in (U_H \times K) \cap (\Gamma \times_h \Gamma') \Rightarrow |t_h(x, y) - 1| \leq \delta/2.$$

Define $U_1^\delta := s(U_H)$. Then:

$$(x, y) \in (U_1^\delta \times K) \cap (\Gamma *_h \Gamma') \Rightarrow (s(x), y) \in (U_H \times K) \cap (\Gamma \times_h \Gamma') \Rightarrow |t_h(s(x), y) - 1| \leq \delta/2.$$

But from Lemma 4.11 $t_h(s(x), y) = \frac{1}{t_h(x, s'(h_a^L(x))y)}$, $a := e'_L(y)$, so finally we have:

$$|t_h(x, s'(h_a^L(x))y) - 1| = \frac{|t_h(s(x), y) - 1|}{|t_h(s(x), y)|} \leq \delta.$$

■

Lemma 4.20 *For any compact $L \subset \Gamma^0$, there exists sequence $U_{n+1} \subset U_n$ of open subsets of Γ with the following properties:*

a) $\forall n \in \mathbb{N}$, $L \subset U_n$, b) $\overline{U_1}$ is compact, c) $s(U_n) = U_n$, d) $\bigcap_{n \in \mathbb{N}} U_n \subset \Gamma^0$, e) for any open $V \supset \Gamma^0$ there exists N_V such that $U_n \subset V$ for all $n > N_V$.

Proof: We begin with the following observation: for any $a \in \Gamma^0$ there exist neighbourhoods $U_a \subset \overline{U_a} \subset U'_a$ and mappings ϕ_a, ϕ'_a with properties:

a) (U_a, ϕ_a) , (U'_a, ϕ'_a) – are maps submitted to submersion e_L and $\phi_a = \phi'_a|_{U_a}$,
b) $\overline{U'_a}$ is compact,

- c) $s(U_a) = U_a$,
d) $\phi_a : U_a \longrightarrow I^m \times I^n$ - where $I^k :=]-1, 1[^k$,
e) $\phi(U_a \cap \Gamma^0) \subset \{0\} \times I^m$.

From the open covering of L $\{U_a, a \in L\}$ (U_a - as above) we choose finite: $U_1 := U_{a_1} \cup \dots \cup U_{a_m}$, then we define for $k \in N$: $\tilde{U}_a^k := \phi_a^{-1}(\phi_a(U_a) \cap (]-\frac{1}{k}, \frac{1}{k}[^m \times I^n))$, $U_a^k := s(\tilde{U}_a^k) \cap \tilde{U}_a^k$ and $U_n := U_{a_1}^n \cup \dots \cup U_{a_m}^n$. Then the family $\{U_n\}_{n \in N}$ has the desired properties.

■

Let $L \subset \Gamma^0$ be compact. Let $\{h_n\}_{n \in N}$ be a sequence of smooth functions on Γ satisfying conditions:

- a) $0 \leq h_n \leq 1$, b) $\text{supp } h_n \subset U^n$, c) $h_n \equiv 1$ on L , d) $h_{n+1} \leq h_n$.

Then the functions $\Gamma^0 \ni a \mapsto \int_{F_l(a)} h_n(x) \lambda_0^2(x)$ are smooth, nonnegative and separated from 0 on L . So one can find g_n - smooth, nonnegative functions on Γ^0 such that: $f_n(x) := h_n(x)g_n(e_L(x))$ is smooth, nonnegative function with compact support contained in U^n and $\int_{F_l(a)} f_n(x) \lambda_0^2(x) = 1$ for $a \in L$. Let us define $\omega_n(x) := f_n(x) \omega_0(x)$.

Proof of the proposition: Let $\omega' \in \mathcal{A}(\Gamma')$ $\omega' = f' \omega'_0$ has the support contained in K and let U_K be as in the lemma 4.17. Then for any $\omega \in \mathcal{A}(\Gamma)$ with support in U_K , $\hat{h}(\omega)(\omega')$ has support contained in H - the fixed compact subset of Γ' . Take $\delta > 0$ then from lemmas 4.18 and 4.19 we get U^δ and U_1^δ - open neighbourhoods of Γ^0 in Γ such that for any $(x, z) \in ((U^\delta \cap U_1^\delta) \times H) \cap (\Gamma *_h \Gamma')$ we have: $|f'(s'(h_a^L(x))z) - f'(z)| \leq \delta$ and $|t_h(x, s'(h_a^L(x))z) - 1| \leq \delta$. Let $U := U_K \cap U^\delta \cap U_1^\delta$ and $L := f_h(e'_L(H))$. Let $\omega_n = f_n \omega_0$ be as above. Then for $n > N_0$ support of ω_n is contained in U . Then the support of $\hat{h}(\omega_n)\omega'$ is contained in H .

For $z \in H$ we have: $\hat{h}(\omega_n)\omega' = (f_n *_h f') \omega'_0$ and $f_n *_h f'(z) := \int_{F_l(a)} \lambda_0^2(x) f_n(x) t_h(x, yz) f'(yz) a := f_h(e'_L(z))$ and we put for simplicity of the notation $y := s'(h_a^L(x))$.

$$\begin{aligned} |f_n *_h f'(z) - f'(z)| &= \left| \int_{F_l(a)} \lambda_0^2(x) f_n(x) t_h(x, yz) f'(yz) - f'(z) \right| = \\ &= \left| \int_{F_l(a)} \lambda_0^2(x) [f_n(x) (t_h(x, yz) f'(yz) - f'(z))] \right| \leq \int_{F_l(a)} \lambda_0^2(x) |f_n(x)| |t_h(x, yz) f'(yz) - f'(z)| = \\ &= \int_{F_l(a)} \lambda_0^2(x) |f_n(x)| |f'(yz) - f'(z)| + \int_{F_l(a)} \lambda_0^2(x) |f_n(x)| |t_h(x, yz) - 1| |f'(yz)| \leq \\ &\leq \delta \left(\int_{F_l(a)} \lambda_0^2(x) |f_n(x)| + \int_{F_l(a)} \lambda_0^2(x) |f_n(x)| |f'(yz)| \right) \leq \delta(1 + M), \text{ for } M := \sup |f'|. \end{aligned}$$

In this way $\sup |f_n *_h f' - f'| \leq \delta(1 + M)$ for $n > N_0$.

Let g be a smooth function on Γ' with compact support $0 \leq g(x) \leq 1$ and $g|_H = 1$.

Then

$$\begin{aligned} \|\hat{h}(\omega_n)\omega' - \omega'\|_L &= \sup_{b \in \Gamma'^0} \int_{F_l(b)} \lambda_0'^2(z) |f_n *_h f'(z) - f'(z)| \leq \\ &\leq \sup_{b \in \Gamma'^0} \int_{F_l(b)} \lambda_0'^2(z) g(z) |f_n *_h f'(z) - f'(z)| \leq \\ &\delta(1 + M) \sup_{b \in \Gamma'^0} \int_{F_l(b)} \lambda_0'^2(z) g(z) \leq \delta(1 + M) M_g, \text{ where } M_g := \sup_{b \in \Gamma'^0} \int_{F_l(b)} \lambda_0'^2(z) g(z). \end{aligned}$$

In the same way we have $\|\hat{h}(\omega_n)\omega' - \omega'\|_R \leq \delta(1 + M) m_g$, where $m_g := \sup_{b \in \Gamma'^0} \int_{F_r(b)} \rho_0'^2(z) g(z)$. This proves that $\lim_{n \rightarrow \infty} \hat{h}(\omega_n)\omega' = \omega'$.

■

Corollary 4.21 *The action of morphism on bidensities is nondegenerate. Indeed, if $\hat{h}(\omega)\omega' = 0$ for any ω then taking ω_n as above we have: $0 = \lim_{n \rightarrow \infty} \hat{h}(\omega_n)\omega' = \omega'$.*

Corollary 4.22 *The representation π_h associated with morphism h is nondegenerate.*

Proof: Choose $\Psi_0 = \rho'_0 \otimes \nu_0$ and let $\Psi := \psi \Psi_0$ be a smooth half density on Γ' with compact support. Let ω_n be as above. Then $\sup |f_n *_h \psi - \psi| \leq \delta(1+M)$ for $n > N_0$ and $M := \sup |\psi|$. Let g be a smooth function on Γ' with compact support $0 \leq g(x) \leq 1$ and $g|_H = 1$.

$$\begin{aligned} \|\pi_h(\omega_n)\Psi - \Psi\|^2 &= \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(x) |f_n *_h \psi - \psi|^2 \leq \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(x) |f_n *_h \psi - \psi|^2 |g(x)|^2 \leq \\ &\leq \delta^2(1+M)^2 \int_{E'} \nu_0^2(a) \int_{\Gamma'_a} \rho_0'^2(x) |g(x)|^2 = \delta^2(1+m)^2 \|\Psi_g\|^2, \end{aligned}$$

where $\Psi_g := g \Psi_0$ and $n > N_0$.

So for any Ψ -smooth half density on Γ' with compact support there exists a sequence $\{\omega_n\}, \omega_n \in \mathcal{A}(\Gamma)$ such that $\lim_{n \rightarrow \infty} \pi_h(\omega_n)\Psi = \Psi$.

Let $\Phi \in L^2(\Gamma')$ be such that $\pi_h(\omega)\Phi = 0$ for any $\omega \in \mathcal{A}(\Gamma)$. Then for any $\Psi \in L^2(\Gamma')$ -smooth with compact support we have $(\Psi, \Phi) = \lim(\pi_h(\omega_n)\Psi, \Phi) = \lim(\Phi, \pi_h(\omega_n^*)\Psi) = 0$. So $\Phi = 0$ and this is nondegeneracy condition.

Remark 4.23 *Note that in the statements above, we don't claim that $\hat{h}\omega_n$ is an approximate identity for $(\mathcal{A}(\Gamma), \|\cdot\|)$, since ω_n depend from chosen ω' . Also $\pi_h(\omega_n)$ does not converge strongly or weakly to identity on $L^2(\Gamma')$.*

Proof of the Lemma.4.4

A. The set $\Gamma \times_h \Gamma' \times_k \Gamma''_a := \{(x, y, z) \in \Gamma \times \Gamma' \times \Gamma''_a : e_R(x) = f_h(e_L(y)), e_R(y) = f_k(e_L(z))\}$ is a submanifold in $(\Gamma \times_h \Gamma') \times \Gamma''_a$ and in $\Gamma \times (\Gamma' \times_k \Gamma''_a)$.

Indeed $\Gamma \times_h \Gamma' \times_k \Gamma''_a = (e'_R \pi_2 \times f_k e''_L)^{-1}(\text{diag}(E' \times E'))$ where $\pi_2 : \Gamma \times_h \Gamma' \rightarrow \Gamma'$ is a projection onto the second factor. It is easy to see that $(e'_R \pi_2 \times f_k e''_L) \lhd \text{diag}(E' \times E')$. We also have: $\Gamma \times_h \Gamma' \times_k \Gamma''_a = (e_R \times f_h e'_L \pi_1)^{-1}(\text{diag}(E \times E))$ where $\pi_1 : \Gamma' \times_k \Gamma''_a \rightarrow \Gamma'$ is a projection onto the first factor and $(e_R \times f_h e'_L \pi_1) \lhd \text{diag}(E \times E)$.

The mappings: $\Gamma \times_h \Gamma' \times_k \Gamma''_a \ni (x, y, z) \mapsto z \in \Gamma''_a$ and $\Gamma \times_h \Gamma' \times_k \Gamma''_a \ni (x, y, z) \mapsto (y, z) \in \Gamma' \times_k \Gamma''_a$ are surjective submersions — this is due to the fact that for any morphism $h : \Gamma \rightarrow \Gamma'$ the mappings $\Gamma \times_h \Gamma' \ni (x, y) \mapsto y \in \Gamma'$ and $\Gamma \times_h \Gamma' \ni (x, y) \mapsto y \in \Gamma'_a$ are surjective submersions.

From this it follows that we have the following isomorphisms:

$$\begin{aligned} i_1 : \Omega_R^{1/2}(x) \otimes \Omega_R^{1/2}(y) &\longrightarrow \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_b) \text{ where } b := e'_R(y) . \\ i_2 : \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_b) \otimes \Omega_R^{1/2}(z) &\longrightarrow \Omega^{1/2} T_{(x,y,z)}(\Gamma \times_h \Gamma' \times_k \Gamma''_a) . \\ i_3 : \Omega_R^{1/2}(y) \otimes \Omega_R^{1/2}(z) &\longrightarrow \Omega^{1/2} T_{(y,z)}(\Gamma' \times_k \Gamma''_a) . \\ i_4 : \Omega_R^{1/2}(x) \otimes \Omega^{1/2} T_{(y,z)}(\Gamma' \times_k \Gamma''_a) &\longrightarrow \Omega^{1/2} T_{(x,y,z)}(\Gamma \times_h \Gamma' \times_k \Gamma''_a) . \end{aligned}$$

Lemma 4.24 $i_2(i_1 \otimes id) = i_4(id \otimes i_3)$

Proof: Let $V := T_{(x,y,z)}(\Gamma \times_h \Gamma' \times_k \Gamma''_a)$. Let also choose B_x, B_y such that $T_x \Gamma = T_x^r \Gamma \oplus B_x$ and $T_y \Gamma' = T_y^r \Gamma' \oplus B_y$. Define: $V_1 := \{(\dot{x}, \dot{y}, \dot{z}) \in V : \dot{x} \in B_x, \dot{y} \in B_y, \dot{z} \in T_z^R \Gamma''\}$ and $W_1 := \{(\dot{x}, \dot{y}, 0) \in V : \dot{y} \in T_y^R \Gamma'\}$. It is clear that $V = V_1 \oplus W_1$, W_1 is a kernel of the projection onto Γ''_a and W_1 is isomorphic to $T_{(x,y)}(\Gamma \times_h \Gamma'_b)$ — the isomorphism is given by (tangent to) projection $\pi_{12} : (x, y, z) \mapsto (x, y)$. The isomorphism i_2 is now given by: $i_2(\omega_{xy} \otimes \rho_z)(v_1 \wedge w_1) := \omega_{xy}(\pi_{12} w_1) \rho_z(\pi_3 v_1)$, where $\omega_{xy} \in \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_b)$, $\rho_z \in \Omega^{1/2} T_z^R \Gamma''$, $v_1 \in \Lambda^{max} V_1$, $w_1 \in \Lambda^{max} W_1$ and π_3 is the projection onto the third factor.

Let $V_2 := \{(\dot{x}, \dot{y}, 0) \in W_1 : \dot{x} \in B_x\}$ and $V_3 := \{(\dot{x}, 0, 0) \in W_1 : \dot{x} \in T_x^r \Gamma\}$. We have $W_1 = V_2 \oplus V_3$ and the isomorphism i_1 is given by $i_1(\rho_x \otimes \rho_y)(v_2 \wedge v_3) := \rho_x(\pi_1 v_3) \rho_y(\pi_2 v_2)$ with the obvious notation. So we have $i_2(i_1 \otimes id)(\rho_x \otimes \rho_y \otimes \rho_z)(v_1 \wedge v_2 \wedge v_3) = \rho_x(\pi_1 v_3) \rho_y(\pi_2 v_2) \rho_z(\pi_3 v_1)$.

It is clear that V_3 is the kernel of the projection $\pi_{23} : (x, y, z) \mapsto (y, z)$. So i_4 is given by: $i_4(\rho_x \otimes \omega_{yz})(v_1 \wedge v_2 \wedge v_3) := \rho_x(\pi_1 v_3) \omega_{yz}(\pi_{23} v_2 \wedge \pi_{23} v_1)$. Also $i_3(\rho_y \otimes \rho_z)(\pi_{23} v_2 \wedge \pi_{23} v_1) := \rho_y(\pi_2 \pi_{23} v_2 \wedge \pi_3 \pi_{23} v_1) = \rho_y(\pi_2 v_2) \rho_z(\pi_3 v_1)$. And this is desired equality. \blacksquare

B. The set $\Gamma *_h \Gamma' \times_k \Gamma''_a := \{(x, y', z) \in \Gamma \times \Gamma' \times \Gamma''_a : e_L(x) = f_h(e_L(y')), e_R(y') = f_k(e_L(z))\}$ is a submanifold in $\Gamma \times (\Gamma' \times_k \Gamma''_a)$ and in $(\Gamma *_h \Gamma') \times \Gamma''_a$.

The argument is as above. Write $\Gamma *_h \Gamma' \times_k \Gamma''_a = (e_L \times f_h e'_L \pi_1)^{-1}(\text{diag}(E \times E)) = (e'_R \pi_2 \times f_k e''_L)^{-1}(\text{diag}(E' \times E'))$ and use the transversality.

The mappings: $\Gamma *_h \Gamma' \times_k \Gamma''_a \ni (x, y', z) \mapsto z \in \Gamma''_a$ and $\Gamma *_h \Gamma' \times_k \Gamma''_a \ni (x, y', z) \mapsto (y', z) \in \Gamma' \times_k \Gamma''_a$ are surjective submersions.

As above this provides us with the following isomorphisms:

$$\begin{aligned} i_5 &: \Omega_L^{1/2}(x) \otimes \Omega_R^{1/2}(y') \longrightarrow \Omega^{1/2}T_{(x,y')}(\Gamma *_h \Gamma'_b). \\ i_6 &: \Omega^{1/2}T_{(x,y')}(\Gamma *_h \Gamma'_b) \otimes \Omega_R^{1/2}(z) \longrightarrow \Omega^{1/2}T_{(x,y',z)}(\Gamma *_h \Gamma' \times_k \Gamma''_a). \\ i_7 &: \Omega_R^{1/2}(y') \otimes \Omega_R^{1/2}(z) \longrightarrow \Omega^{1/2}T_{(y',z)}(\Gamma' \times_k \Gamma''_a). \\ i_8 &: \Omega_L^{1/2}(x) \otimes \Omega^{1/2}T_{(y',z)}(\Gamma' \times_k \Gamma''_a) \longrightarrow \Omega^{1/2}T_{(x,y',z)}(\Gamma *_h \Gamma' \times_k \Gamma''_a). \end{aligned}$$

Lemma 4.25 $i_8(id \otimes i_7) = i_6(i_5 \otimes id)$

Proof: As above choose $B_x, B_{y'}$ such that: $T_x \Gamma = T_x^l \Gamma \oplus B_x$ and $T_{y'} \Gamma' = T_{y'}^r \Gamma' \oplus B_{y'}$. Then $V := T_{(x,y',z)}(\Gamma *_h \Gamma' \times_k \Gamma''_a)$ has the decomposition: $V = V_1 \oplus V_2 \oplus V_3$ for $V_1 := \{(\dot{x}, \dot{y}', \dot{z}) \in V : \dot{x} \in B_x, \dot{y}' \in B_{y'}, \dot{z} \in T_z^r \Gamma''\}$, $V_2 := \{(\dot{x}, \dot{y}, 0) \in V : \dot{x} \in B_x, \dot{y}' \in T_{y'}^r \Gamma'\}$ and $V_3 := \{(\dot{x}, 0, 0) \in V : \dot{x} \in T_x^l \Gamma\}$. And the lemma follows in the same way as the previous one.

■

C. The set $\Gamma *_h \Gamma' *_k \Gamma''_a := \{(x, y', z') \in \Gamma \times \Gamma' \times \Gamma''_a : e_L(x) = f_h(e_L(y')), e_L(y') = f_k(e_L(z'))\}$ is a submanifold in $\Gamma \times \Gamma' \times \Gamma''_a$.

The mappings: $\Gamma *_h \Gamma' *_k \Gamma''_a \ni (x, y', z') \mapsto (y', z') \in \Gamma' *_k \Gamma''_a$ and $\Gamma *_h \Gamma' *_k \Gamma''_a \ni (x, y', z') \mapsto (x, z') \in \Gamma *_k \Gamma''_a$ are surjective submersions. These fact can be seen in the same way as above. Again we have isomorphisms:

$$\begin{aligned} i_9 &: \Omega_L^{1/2}(y') \otimes \Omega_R^{1/2}(z') \longrightarrow \Omega^{1/2}T_{(y',z')}(\Gamma *_k \Gamma''_a). \\ i_{10} &: \Omega_L^{1/2}(x) \otimes \Omega^{1/2}T_{(y',z')}(\Gamma *_k \Gamma''_a) \longrightarrow \Omega^{1/2}T_{(x,y',z')}(\Gamma *_h \Gamma' *_k \Gamma''_a). \\ i_{11} &: \Omega_L^{1/2}(x) \otimes \Omega_R^{1/2}(z') \longrightarrow \Omega^{1/2}T_{(x,z')}(\Gamma *_k \Gamma''_a). \\ i_{12} &: \Omega^{1/2}T_{(x,z')}(\Gamma *_k \Gamma''_a) \otimes \Omega_L^{1/2}(y') \longrightarrow \Omega^{1/2}T_{(x,y',z')}(\Gamma *_h \Gamma' *_k \Gamma''_a). \end{aligned}$$

Lemma 4.26 $i_{10}(id \otimes i_9) = i_{12}(i_{11} \otimes id)(id \otimes \sim)$, where $\sim: \Omega_L^{1/2}(y') \otimes \Omega_R^{1/2}(z') \longrightarrow \Omega_R^{1/2}(z') \otimes \Omega_L^{1/2}(y')$ is the flip.

Proof: The proof is based on the same arguments as above.

■

D. $\Gamma \times_h \Gamma' *_k \Gamma''_a := \{(x, y, z'') \in \Gamma \times \Gamma' \times \Gamma''_a : e_R(x) = f_h(e_L(y)), e_L(y) = f_k(e_L(z''))\}$ – this is submanifold in $\{(x, y, z'') \in \Gamma \times \Gamma' \times \Gamma''_a : (x, z'') \in \Gamma \times_{kh} \Gamma''_a\}$ and in $\Gamma \times (\Gamma' *_k \Gamma''_a)$.

The mappings: $\Gamma \times_h \Gamma' *_k \Gamma''_a \ni (x, y, z'') \mapsto (y, z'') \in \Gamma' *_k \Gamma''_a$ and $\Gamma \times_h \Gamma' *_k \Gamma''_a \ni (x, y, z'') \mapsto (x, z'') \in \Gamma \times_{kh} \Gamma''_a$ are surjective submersions. And again we have isomorphisms:

$$\begin{aligned} i_{13} &: \Omega_R^{1/2}(x) \otimes \Omega^{1/2}T_{(y,z'')}(\Gamma' *_k \Gamma''_a) \longrightarrow \Omega^{1/2}T_{(x,y,z'')}(\Gamma \times_h \Gamma' *_k \Gamma''_a). \\ i_{14} &: \Omega_L^{1/2}(y) \otimes \Omega_R^{1/2}(z'') \longrightarrow \Omega^{1/2}T_{(y,z'')}(\Gamma' *_k \Gamma''_a). \\ i_{15} &: \Omega_L^{1/2}(y) \otimes \Omega^{1/2}T_{(x,z'')}(\Gamma \times_{kh} \Gamma''_a) \longrightarrow \Omega^{1/2}T_{(x,y,z'')}(\Gamma \times_h \Gamma' *_k \Gamma''_a). \\ i_{16} &: \Omega_R^{1/2}(x) \otimes \Omega_R^{1/2}(z'') \longrightarrow \Omega^{1/2}T_{(x,z'')}(\Gamma \times_{kh} \Gamma''_a). \end{aligned}$$

Lemma 4.27 $i_{13}(id \otimes i_{14}) = i_{15}(id \otimes i_{16})(\sim \otimes id)$

Proof: As above.

■

E. The mapping $(t_h \times id) : \Gamma \times_h \Gamma' \times_k \Gamma''_a \longrightarrow \Gamma *_h \Gamma' \times_k \Gamma''_a$ is a diffeomorphism.

The mapping $(id \times t_k) : \Gamma *_h \Gamma' \times_k \Gamma''_a \longrightarrow \Gamma *_h \Gamma' *_k \Gamma''_a$ is a diffeomorphism.

The mapping $(id \times t_k) : \Gamma \times_h \Gamma' \times_k \Gamma''_a \longrightarrow \Gamma \times_h \Gamma' *_k \Gamma''_a$ is a diffeomorphism.

Define the mapping $\tilde{t} : \Gamma \times_h \Gamma' *_k \Gamma''_a \ni (x, y, z) \mapsto (x, m_h(x, y), m_{kh}(x, z)) \in \Gamma *_h \Gamma' *_k \Gamma''_a$.

Then: $(id \times t_k)(t_h \times id) = \tilde{t}(id \times t_k) : \Gamma \times_h \Gamma' \times_k \Gamma''_a \longrightarrow \Gamma *_h \Gamma' *_k \Gamma''_a$. So \tilde{t} is a diffeomorphism.

F. Let $(x, y, z) \in \Gamma \times_h \Gamma' \times_k \Gamma''_a$ and let y', z', z'' be defined by: $t_h(x, y) = (x, y')$, $t_k(y', z) = (y', z')$, $t_k(y, z) = (y, z'')$. The situation is illustrated on the figure 4.4.

From the definition we have: $t_h i_1(\rho_0(x) \otimes \rho_0(y)) =: t_h(x, y) i_5(\lambda_0(x) \otimes \rho_0(y'))$,

$t_k i_7(\rho_0(y') \otimes \rho_0(z)) =: t_k(y', z) i_9(\lambda_0(y') \otimes \rho_0(z'))$,

$t_k i_3(\rho_0(y) \otimes \rho_0(z)) =: t_k(y, z) i_{14}(\lambda_0(y) \otimes \rho_0(z''))$ and $t_{kh} i_{16}(\rho_0(x) \otimes \rho_0(z'')) =: t_{kh}(x, z'') i_{11}(\lambda_0(x) \otimes \rho_0(z'))$.

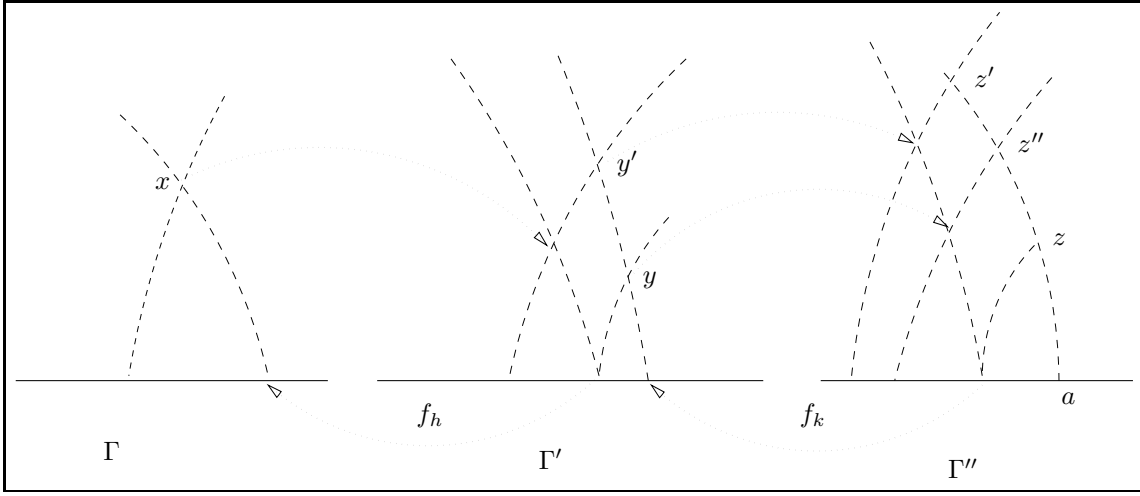


Figure 4.4:

Lemma 4.28 a) $(t_h \times id)i_2(\alpha \otimes \rho_z) = i_6(t_h \alpha \otimes \rho_z)$ b) $(id \times t_k)i_8(\lambda_x \otimes \beta) = i_{10}(\lambda_x \otimes t_k \beta)$
c) $(id \times t_k)i_4(\rho_x \otimes \beta) = i_{13}(\rho_x \otimes t_k \beta)$ d) $\tilde{t}_{i_{15}}(\lambda_y \otimes \beta) = i_{12} \sim (h_{e_L^R(y)}^R(x)\lambda_y \otimes t_{kh}\beta).$

Proof: a) Let $V := T_{(x,y,z)}(\Gamma \times_h \Gamma' \times_k \Gamma''_a)$ and $V_1, W_1 \subset V$ be as in **A**. Then $(t_h \times id)W_1$ is a kernel of the projection $(x, y', z) \mapsto z$. Since $t_h \times id$ is a diffeomorphism, $(t_h \times id)V_1$ is complementary to this kernel. Compute the left hand side:

$$(t_h \times id)i_2(\alpha \otimes \rho_z)((t_h \times id)v_1 \otimes (t_h \times id)w_1) = i_2(\alpha \otimes \rho_z)(v_1 \otimes w_1) = \alpha(\pi_{12}w_1)\rho_z(\pi_3v_1).$$

And the right hand side:

$$\begin{aligned} i_6(t_h \alpha \otimes \rho_z)((t_h \times id)v_1 \otimes (t_h \times id)w_1) &= t_h \alpha(\pi_{12}(t_h \times id)w_1)\rho_z(\pi_3(t_h \times id)v_1) = \\ &= t_h \alpha(t_h \pi_{12}w_1)\rho_z(\pi_3v_1) = \alpha(\pi_{12}w_1)\rho_z(\pi_3v_1). \end{aligned}$$

This proves a). Statements b) and c) can be proved in the same way.

Let us prove d). Let $V := T_{(x,y,z'')}(\Gamma \times_h \Gamma' *_k \Gamma''_a)$, $V_2 := \{(0_x, \dot{y}, 0_{z''}) \in V : \dot{y} \in T_y^l \Gamma'\}$ and V_1 be such that $V = V_1 \oplus V_2$. The isomorphism $i_{15} : \Omega_L^{1/2}(y) \otimes \Omega^{1/2}T_{(x,z'')}(\Gamma \times_{kh} \Gamma''_a) \longrightarrow \Omega^{1/2}T_{(x,y,z'')}(\Gamma \times_h \Gamma' *_k \Gamma''_a)$ is given by: $i_{15}(\lambda_y \otimes \beta)(v_1 \wedge v_2) := \lambda_y(\pi_2 v_2)\beta(\pi_{13}v_1)$.

Let also $\tilde{V} := T_{(x,y',z')}(\Gamma *_h \Gamma' *_k \Gamma''_a)$, $\tilde{V}_2 := \{(0_x, \dot{y}', 0_{z'}) \in \tilde{V} : \dot{y}' \in T_{y'}^l \Gamma'\}$ and \tilde{V}_1 be complementary to \tilde{V}_2 . Then $i_{12} : \Omega^{1/2}T_{(x,z')}(\Gamma *_k \Gamma''_a) \otimes \Omega_L^{1/2}(y') \longrightarrow \Omega^{1/2}T_{(x,y',z')}(\Gamma *_h \Gamma' *_k \Gamma''_a)$ is given by: $i_{12}(\beta \otimes \lambda_{y'})(\tilde{v}_1 \wedge \tilde{v}_2) := \beta(\pi_{13}\tilde{v}_1)\lambda_{y'}(\pi_2\tilde{v}_2)$.

We have: $\tilde{t}_{i_{15}}(\lambda_y \otimes \beta)(\tilde{v}_1 \wedge \tilde{v}_2) := \lambda_y(\pi_2 v_2)\beta(\pi_{13}v_1)$.

It is easy to see that $\tilde{t}V_2 = \tilde{V}_2$ so $\tilde{t}V_1$ is complementary to \tilde{V}_2 . From this we have:

$$\begin{aligned} i_{12}(t_{kh}\beta \otimes h_{e_L^R(y)}^R \lambda_y)(\tilde{t}v_1 \wedge \tilde{t}v_2) &= (h_{e_L^R(y)}^R \lambda_y)(\pi_2 \tilde{t}v_2)(t_{kh}\beta)(\pi_{13}\tilde{t}v_1) = \\ &= (h_{e_L^R(y)}^R \lambda_y)(h_{e_L^R(y)}^R \pi_2 v_2)(t_{kh}\beta)(t_{kh}\pi_{13}v_1) = \lambda_y(\pi_2 v_2)\beta(\pi_{13}v_1). \end{aligned}$$

Where we used: $\pi_{13}\tilde{t} = t_{kh}\pi_{13}$ and $\pi_2\tilde{t}v_2 = h_{e_L^R(y)}^R \pi_2 v_2$.

■

Proof of the Lemma 4.4:

$$\begin{aligned} (id \times t_k)(t_h \times id)i_2(i_1 \otimes id)(\rho_0(x) \otimes \rho_0(y) \otimes \rho_0(z)) &= (id \times t_k)i_6(t_h i_1(\rho_0(x) \otimes \rho_0(y)) \otimes \rho_0(z)) = \\ &= t_h(x, y)(id \times t_k)i_6(i_5(\lambda_0(x) \otimes \rho_0(y')) \otimes \rho_0(z)) = t_h(x, y)(id \times t_k)i_8(id \otimes i_7)(\lambda_0(x) \otimes \rho_0(y') \otimes \rho_0(z)) = \\ &= t_h(x, y)i_{10}(\lambda_0(x) \otimes t_k i_7(\rho_0(y') \otimes \rho_0(z))) = t_h(x, y)t_k(y', z)i_{10}(id \times i_9)(\lambda_0(x) \otimes \lambda_0(y') \otimes \rho_0(z')). \end{aligned}$$

From the other side:

$$\tilde{t}(id \times t_k)i_2(i_1 \otimes id)(\rho_0(x) \otimes \rho_0(y) \otimes \rho_0(z)) = \tilde{t}(id \times t_k)i_4(id \otimes i_3)(\rho_0(x) \otimes \rho_0(y) \otimes \rho_0(z)) =$$

$$\begin{aligned}
&= \tilde{t}_{13}(\rho_0(x) \otimes t_k i_3(\rho_0(y) \otimes \rho_0(z))) = t_k(y, z) \tilde{t}_{13}(\rho_0(x) \otimes i_{14}(\lambda_0(y) \otimes \rho_0(z''))) = \\
&= t_k(y, z) \tilde{t}_{15}(\lambda_0(y) \otimes i_{16}(\rho_0(x) \otimes \rho_0(z''))) = t_k(y, z) i_{12} \sim (h_{e'_L(y)}^R(x) \lambda_0(y) \otimes t_{kh} i_{16}(\rho_0(x) \otimes \rho_0(z''))) = \\
&= t_k(y, z) t_{kh}(x, z'') i_{12} \sim (\lambda_0(y') \otimes i_{11}(\lambda_0(x) \otimes \rho_0(z'))) = t_k(y, z) t_{kh}(x, z'') i_{12} (i_{11}(\lambda_0(x) \otimes \rho_0(z') \otimes \lambda_0(y'))) = \\
&= t_k(y, z) t_{kh}(x, z'') i_{12} (i_{11} \otimes id)(id \otimes \sim)(\lambda_0(x) \otimes \lambda_0(y') \otimes \rho_0(z')) = t_k(y, z) t_{kh}(x, z'') i_{10}(id \otimes i_9)(\lambda_0(x) \otimes \lambda_0(y') \otimes \rho_0(z')).
\end{aligned}$$

So $t_h(x, y) t_k(y', z) = t_k(y, z) t_{kh}(x, z'')$.

■

5 C^* -algebra of a differential groupoid.

In this section we define the C^* -algebra of a differential groupoid and show that this correspondence is a covariant functor to the category of C^* -algebras.

Let Γ be a differential groupoid. In the previous section we have established the following facts:

- 1) With any morphism $h : \Gamma \rightarrow \Gamma'$ one can associate two objects: the mapping $\hat{h} : \mathcal{A}(\Gamma) \rightarrow LM\mathcal{A}(\Gamma')$ and the non degenerate representation π_h of the $*$ -algebra $\mathcal{A}(\Gamma)$ in $L^2(\Gamma')$.
- 2) $\omega_3^*(\hat{h}(\omega_1)\omega_2) = (\hat{h}(\omega_1^*)\omega_3)^*\omega_2$ for any $\omega_1 \in \mathcal{A}(\Gamma)$, $\omega_2, \omega_3 \in \mathcal{A}(\Gamma')$.
- 3) If $k : \Gamma' \rightarrow \Gamma''$ is a morphism of differential groupoids. Then:
 - a) $\hat{k}(\hat{h}(\omega_1)\omega_2)\omega_3 = \hat{k}\hat{h}(\omega_1)(\hat{k}(\omega_2)\omega_3)$, $\omega_1 \in \mathcal{A}(\Gamma)$, $\omega_2 \in \mathcal{A}(\Gamma')$, $\omega_3 \in \mathcal{A}(\Gamma'')$.
 - b) $\pi_k(\hat{h}(\omega_1)\omega_2)\psi = \pi_{kh}(\omega_1)(\pi_k(\omega_2)\psi)$, for $\omega_1 \in \mathcal{A}(\Gamma)$, $\omega_2 \in \mathcal{A}(\Gamma')$, $\psi \in L^2(\Gamma'')$.
- 4) We can choose norm on $\mathcal{A}(\Gamma)$ which agrees with $*$ -algebra structure, such that: $\|\pi_h(\omega)\| \leq \|\omega\|$.
- 5) Exists morphism l from Γ to the pair groupoid $\Gamma \times \Gamma$ such that π_l is a faithful representation of $\mathcal{A}(\Gamma)$.

From 4) and 5) follows that the following definition is meaningful:

Definition 5.1 The C^* -algebra of a differential groupoid Γ is a completion of $\mathcal{A}(\Gamma)$ with respect to the norm: $\|\omega\| := \sup_h \|\pi_h(\omega)\|$, where the supremum is taken over all morphisms $h : \Gamma \rightarrow \Gamma'$.

Proposition 5.2 For any morphism $h : \Gamma \rightarrow \Gamma'$, π_h extends to a nondegenerate representation of $C^*(\Gamma)$ and \hat{h} extends to $C^*(h) \in Mor(C^*(\Gamma), C^*(\Gamma'))$.

Proof: Let us recall (cf. appendix E) that φ is a morphism of C^* -algebras A, B iff $\varphi : A \rightarrow M(B)$ is a $*$ -algebra homomorphism and the set $\varphi(A)B$ is dense in B . Let $k : \Gamma' \rightarrow \Gamma''$ be any morphism. From prop. 4.13 one has:

$$\|\pi_k(\hat{h}(\omega_1)\omega_2)\| \leq \|\pi_{kh}(\omega_1)\| \|\pi_k(\omega_2)\| \leq \|\omega_1\|_{C^*(\Gamma)} \|\omega_2\|_{C^*(\Gamma')},$$

so also: $\|\hat{h}(\omega_1)\omega_2\|_{C^*(\Gamma')} \leq \|\omega_1\|_{C^*(\Gamma)} \|\omega_2\|_{C^*(\Gamma')}$.

Since $\mathcal{A}(\Gamma')$ is dense in $C^*(\Gamma')$ $\hat{h}(\omega_1)$ can be extended to a continuous linear mapping on $C^*(\Gamma')$ and by the density of $\mathcal{A}(\Gamma)$ in $C^*(\Gamma)$ \hat{h} defines a bounded algebra homomorphism $C^*(h) : C^*(\Gamma) \rightarrow BC^*(\Gamma')$.

From 4.10 : $\omega_3^*(\hat{h}(\omega_1)\omega_2) = (\hat{h}(\omega_1^*)\omega_3)^*\omega_2$ for any $\omega_1 \in \mathcal{A}(\Gamma)$, $\omega_2, \omega_3 \in \mathcal{A}(\Gamma')$.

By density of $\mathcal{A}(\Gamma')$ and continuity this equality extends to $\omega_2, \omega_3 \in C^*(\Gamma')$. So for any $\omega \in \mathcal{A}(\Gamma)$ $\hat{h}(\omega)$ has the hermitian conjugate (see appendix E) equal to $\hat{h}(\omega^*)$ and $C^*(h)(\omega) \in MC^*(\Gamma')$. It follows that $C^*(h)$ is a continuous $*$ -algebra homomorphism $\mathcal{A}(\Gamma) \rightarrow MC^*(\Gamma')$. By continuity it is also true for $\omega \in C^*(\Gamma)$.

From the Prop. 4.16 for any $\omega' \in \mathcal{A}(\Gamma')$ one can find $\omega_n \in \mathcal{A}(\Gamma)$ such that $\|\hat{h}(\omega_n)\omega' - \omega'\|_0 \rightarrow 0$ for $*$ -norm on $\mathcal{A}(\Gamma')$ given by some ω'_0 . So also $\|\hat{h}(\omega_n)\omega' - \omega'\|_{C^*(\Gamma')} \rightarrow 0$ and $\mathcal{A}(\Gamma') \subset \overline{C^*(h)(\mathcal{A}(\Gamma))\mathcal{A}(\Gamma')}$. Since $\mathcal{A}(\Gamma')$ is dense in $C^*(\Gamma')$ this is nondegeneracy condition.

In this way $C^*(h) \in Mor(C^*(\Gamma), C^*(\Gamma'))$.

From the definition $\|\pi_h(\omega)\| \leq \|\omega\|_{C^*(\Gamma)}$ for $\omega \in \mathcal{A}(\Gamma)$. So π_h defines representation of $C^*(\Gamma)$. The nondegeneracy condition is just the Col. 4.22.

■

The next proposition shows the functoriality of C^* .

Proposition 5.3 $C^*(kh) = C^*(k)C^*(h)$.

Proof: Let $\phi_1 := C^*(h)$, $\phi_2 := C^*(k)$, $\phi_3 := C^*(kh)$. Let $\hat{\phi}_2 : MC^*(\Gamma') \rightarrow MC^*(\Gamma'')$ denote the unique extension of ϕ_2 (see appendix E). We have to show that $\phi_3 = \hat{\phi}_2\phi_1$. In fact it is enough to show the equality

for all $\omega_1 \in \mathcal{A}(\Gamma)$: $\phi_3(\omega_1) = \hat{\phi}_2\phi_1(\omega_1)$. Since $\hat{k}(\mathcal{A}(\Gamma'))\mathcal{A}(\Gamma'')$ is dense in $C^*(\Gamma'')$ it is enough to show that: $\hat{k}\hat{h}(\omega_1)\hat{k}(\omega_2)\omega_3 = \hat{\phi}_2\hat{h}(\omega_1)\hat{k}(\omega_2)\omega_3 = \hat{k}(\hat{h}(\omega_1)\omega_2)\omega_3$. But this is Prop. 4.3.

■

The definition of the C^* norm given above is rather abstract and obtained C^* algebra seems untreatable. However, as is shown below, we can restrict ourself to a smaller class of morphisms—namely to the morphisms to the pair groupoids.

Let $h : \Gamma \rightarrow \Gamma'$ be a morphism of differential groupoids and let $l' : \Gamma' \rightarrow \Gamma' \times \Gamma'$ be a left regular representation as defined in 2.11 f) ($\Gamma' \times \Gamma'$ is a pair groupoid, not the product of groupoids). Then $\tilde{h} := l'h$ is a morphism from Γ to the pair groupoid $\Gamma' \times \Gamma'$. Then it is easy to see that: $f_{\tilde{h}} = f_h e'_L$, $\Gamma \times_{\tilde{h}} (\Gamma' \times \Gamma') = (\Gamma \times_h \Gamma') \times \Gamma'$, $\Gamma *_{\tilde{h}} (\Gamma \times \Gamma') = (\Gamma *_h \Gamma') \times \Gamma'$, $m_{\tilde{h}} = m_h \times id$ and $t_{\tilde{h}} = t_h \times id$. Taking into account these formulas and equality $L^2(\Gamma' \times \Gamma') = L^2(\Gamma') \otimes L^2(\Gamma')$ one can show that: $\pi_{\tilde{h}}(\omega) = \pi_h(\omega) \otimes I$ so (semi)norm coming from π_h and $\pi_{\tilde{h}}$ are equal. Since all manifolds are second countable, our C^* algebra is separable.

Examples 5.4 a) *Pair groupoids.* Let $\Gamma := X \times X$, $\Gamma' := Y \times Y$ be pair groupoids. Due to the structure of morphisms in this case as explained in Example 3.6b) we can assume that $Y = X \times Z$ for some manifold Z and $h : \Gamma \rightarrow \Gamma'$ is given by $Gr(h) := \{(x, z, x', z'; x, x') : x, x' \in X, z \in Z\}$. Choose φ_0 —smooth, real, non vanishing half density on X . Since $T_{(x, x')}^l \Gamma = T_{x'} X$ and $T_{(x, x')}^r \Gamma = T_x X$ it defines λ_0 by the formula $\lambda_0(x, x') = \varphi_0(x')$. Then the corresponding ω_0 is given by $\omega_0(x, x') = \varphi_0(x') \otimes \varphi_0(x)$. Choose also μ_0 —smooth, real, non vanishing half density on Z . Then $\psi_0(x, z) := \varphi_0(x) \otimes \mu_0(z)$ is smooth, real, non vanishing half density on Y . Again $\tilde{\psi}_0(y_1, y_2) := \psi_0(y_1) \otimes \psi_0(y_2)$ is smooth, real, non vanishing half density on Γ' , moreover $\tilde{\psi}$ defines right invariant, half density on Γ' along the right fibers, so this is decomposition of $\tilde{\psi}_0$ into the form $\tilde{\psi}_0 = \rho_0 \otimes \nu_0 - \nu_0 = \psi_0$ is half density on the set of identities of Γ' . It is easy to see that:

$$\begin{aligned} \Gamma \times_h \Gamma' &= \{(x, x'; x', z', x'', z'') : x, x', x'' \in X, z', z'' \in Z\}, \\ \Gamma *_h \Gamma' &= \{(x, x'; x, z, x'', z'') : x, x', x'' \in X, z, z'' \in Z\}, \\ t_h(x, x'; x', z', x'', z'') &= (x, x'; x, z', x'', z''). \end{aligned}$$

Also a short computation shows that with the above choice of ω_0, ψ_0 the formula function t_h is constant and equal to 1. So for $\omega = f\omega_0$, $\Psi = \psi\tilde{\psi}_0$ the representation associated with the morphism h is given by:

$$(\pi_h(\omega)\Psi)(y_1, y_2) = \left[\int_X \varphi_0^2(x) f(x_1, x) \psi(y, y_2) \right] \tilde{\psi}_0(y_1, y_2),$$

where $y := (x, z_1)$, $y_1 := (x_1, z_1)$.

We have $L^2(\Gamma') = L^2(Y \times Y) = L^2(Y) \otimes L^2(Y)$ and from the formula it is clear that $\pi_h = \pi_1 \otimes I$ for

$$(\pi_1(\omega)\Psi)(y_1) := \left[\int_X \varphi_0^2(x) f(x_1, x) \psi(y) \right] \psi_0(y_1)$$

for $\Psi = \psi\psi_0$ —smooth, half density on Y with compact support. Again since $Y = X \times Z$ this representation is of the form $\pi_1 = \pi_0 \otimes I$ for

$$(\pi_0(\omega)\Phi)(x_1) := \left[\int_X \varphi_0^2(x) f(x_1, x) \phi(x) \right] \varphi_0(x_1),$$

where $\Phi := \phi\varphi_0$ is smooth, half density on X with compact support.

In this way we have shown, that for pair groupoids the C^* norm on $\mathcal{A}(\Gamma)$ is equal to the norm coming from the left regular representation. The completion of $\mathcal{A}(\Gamma)$ in this norm is the algebra of compact operators.

Groups. Let $\Gamma = G$ be a Lie group. $\mathcal{A}(G)$ is by definition $*$ -algebra of compactly supported, smooth, complex densities on G . For any U -strongly continuous, unitary representation of G on the Hilbert space H the formula

$$(x, \pi_U(\nu)x) := \int_G \nu(g)(x, U(g)x), \nu \in \mathcal{A}(G), x \in H$$

defines nondegenerate $*$ -representation of $\mathcal{A}(G)$. $C^*(G)$ is a completion of $\mathcal{A}(G)$ with respect to the norm $\|\nu\| := \sup \pi_U \|\pi_U(\nu)\|$.

Since for a group, left and right fibers are equal it is clear that any $\omega \in \mathcal{A}(\Gamma)$ is a density on G by an assignment: $\mathcal{A}(\Gamma) \ni \lambda \otimes \rho \mapsto \lambda\rho \in \mathcal{A}(G)$. Let X be a manifold and $\Gamma' := X \times X$ corresponding pair groupoid. By a slight modification of arguments used in Example. 3.6 a) morphisms $h : \Gamma \rightarrow \Gamma'$ are in one to one correspondance with smooth actions of G on X : $Gr(h) := \{(gx, x; g) : x \in X, g \in G\}$. It is easy to see that $\Gamma \times_h \Gamma' = \Gamma *_h \Gamma' = \{(g; x_1, x_2) : x_1, x_2 \in X, g \in G\}$ and $t_h(g; x_1, x_2) = (g; gx_1, x_2)$. Moreover $T_{(g; x, x_0)}(\Gamma \times_h \Gamma'_x) = T_g G \oplus T_x X$. Choose λ_0 – real, smooth, non vanishing left invariant half density on G and ψ_0 –smooth, real, non vanishing half density on X . Then a short computations show that $t_h(g; g^{-1}x, x_0) = \frac{\rho_0}{\lambda_0}(g) \frac{g\psi_0}{\psi_0}(x)$ and for $\omega = f\lambda_0 \otimes \rho_0$, $\Psi = \psi\psi_0 \otimes \psi_0$ we have:

$$(\pi_h(\omega)\Psi)(x_1, x_2) = \left[\int_G \lambda_0^2(g) f(g) t_h(g; g^{-1}x_1, x_2) \psi(g^{-1}x_1, x_2) \right] \psi_0(x_1) \otimes \psi_0(x_2).$$

Since $L^2(\Gamma') = L^2(X) \otimes L^2(X)$ this representation is of form $\pi_h(\omega) = \tilde{\pi}_h(\omega) \otimes I$ for

$$(\tilde{\pi}_h(\omega)\Psi)(x) := \left[\int_G \lambda_0^2(g) f(g) \frac{\rho_0}{\lambda_0}(g) \frac{g\psi_0}{\psi_0}(x) \psi(g^{-1}x) \right] \psi_0(x),$$

where $\Psi = \psi\psi_0$ is a smooth half density on X with compact support.

From the other side, the action of G on X defines strongly continuous unitary representation of G on $L^2(X)$ by the formula: $U_g\Psi := g\Psi$ for Ψ –smooth, compactly supported half density on X . If $\Psi = \psi\psi_0$ then $(U_g\Psi)(x) = \psi(g^{-1}x) \frac{g\psi_0}{\psi_0}(x) \psi_0(x)$. If $\omega = f\lambda_0 \otimes \rho_0$ then $\nu := f\lambda_0\rho_0 = f\frac{\rho_0}{\lambda_0}\lambda_0^2$ and $\tilde{\pi}_h(\omega)\Psi = \pi_U(\nu)\Psi$. In this way $\|\omega\|_{C^*(\Gamma)} \leq \|\nu\|_{C^*(G)}$. It is clear that if π_l is a left regular representation then $\|\pi_l(\omega)\| \leq \|\omega\|_{C^*(\Gamma)}$ so the C^* algebra of a Lie group G viewed as a differential groupoid is something “between” the reduced C^* algebra of G and the algebra $C^*(G)$ where G is treated as locally compact topological group.

Transformation groupoids. Let $\Gamma := G \times X$ be a transformation groupoid. By $C_0(X)$ we denote the C^* algebra of complex, continuous, vanishing at infinity functions on X . The action of G on X induces a strongly continuous homomorphism $\alpha : G \ni g \mapsto \alpha_g \in Aut(C_0(X))$, where $Aut(C_0(X))$ is a group of *-isomorphism of $C_0(X)$, namely $(\alpha_g f)(x) := f(g^{-1}x)$. So $(G, C_0(X), \alpha)$ is a C^* dynamical system (see appendix). Let Y be a manifold and $\Gamma' := Y \times Y$ the corresponding pair groupoid. By Example 3.6 a) morphisms $h : \Gamma \rightarrow \Gamma'$ are in one to one correspondance with smooth actions $G \times Y \ni (g, y) \mapsto gy \in Y$ together with smooth equivariant mapping $F : Y \rightarrow X$. The graph of h is then equal: $Gr(h) = \{(gy, y; g, F(y)) : y \in Y, g \in G\}$. The action of G on Y induces strongly continuous, unitary representation $G \ni g \mapsto U_g \in B(L^2(Y))$. The mapping F defines nondegenerate representation π of $C_0(X)$ on $L^2(Y)$ by the formula: $(\pi(f)\psi)(y) := f(F(y))\psi(y)$, $f \in C_0(X)$ and ψ –smooth, compactly supported half density on Y . The pair (π, U) is a covariant representation of $(G, C_0(X), \alpha)$. Indeed

$$\begin{aligned} (\pi(\alpha_g f)U_g\psi)(y) &= (\alpha_g f)(F(y))(g\psi)(y) = f(g^{-1}F(y))(g\psi)(y) = \\ &= f(F(g^{-1}y))(g\psi)(y) = (\pi(f))(g^{-1}y)(g\psi)(y) = (U_g\pi(f)\psi)(y). \end{aligned}$$

Now we go back to groupoid Γ . Choose $\mu_0 \neq 0$ –real, half density on $T_e G$. Since $T_{(g,x)}^r \Gamma = T_g G$, it defines right invariant, non vanishing, half density on Γ by the formula $\rho_0(g, x)(v_g) := \mu_0(v_g g^{-1})$ where $v_g \in \Lambda^{max} T_g G = \Lambda^{max} T_{(g,x)}^r \Gamma$. The corresponding left invariant half density is given by $\lambda_0(g, x)(v) = \mu_0(g^{-1}\pi(v))$ where $v \in \Lambda^{max} T_{(g,x)}^l \Gamma$ and $\pi : \Gamma \times X \rightarrow G$ is a projection. Let λ be a corresponding left invariant density on G i.e. $\lambda(g) := g\mu_0^2$ and let Δ be a corresponding modular function. Then for $\omega_1 = f_1\omega_0$, $\omega_2 = f_2\omega_0 \in \mathcal{A}(\Gamma)$ we have:

$$(\omega_1\omega_2)(g, x) = \left[\int_G \lambda(g_1) f_1(g_1, g_1^{-1}gx) f_2(g_1^{-1}g, x) \right] \omega_0(g, x) \text{ and } \omega^*(g, x) = \overline{f(g^{-1}, gx)} \omega_0(g, x).$$

Let $K(G, C_0(X))$ be a *-algebra of compactly supported, continuous functions from G to $C_0(X)$ with the usual structure.(see appendix). We define the mapping $\mathcal{A}(\Gamma) \ni \omega \mapsto \hat{\omega} \in K(G, C_0(X))$ by $\hat{\omega}(g)(x) := \Delta(g)^{-1/2} f(g, g^{-1}x)$ for $\omega = f\omega_0$. Straightforward computations show that this is injective *-homomorphism. Let $h : \Gamma \rightarrow \Gamma'$ be a morphism, $Gr(h) := \{(gy, y; g, F(y))\}$ Choose ψ_0 – smooth, real, non vanishing half density on Y . Then $(\psi_0 \otimes \psi_0)(y_1, y_2) := \psi_0(y_1)\psi_0(y_2)$ is real, smooth, non vanishing half density on Γ' . We have: $\Gamma \times_h \Gamma'_y = \{(g, F(y_1); y_1, y) : g \in G, y_1, y \in Y\}$, $\Gamma *_h \Gamma'_y = \{(g, F(y_1); gy_1, y) : g \in G, y_1, y \in Y\}$

and $t_h(g, F(y_1); y_1, y) = (g, F(y_1), gy_1, y)$. Simple computations show that the function t_h is given by: $t_h(g, F(y); y, y_1) = \Delta(g)^{-1/2} \frac{g\psi_0}{\psi_0}(gy)$. So the representation π_h is given by:

$$(\pi_h(\omega)\Psi)(y_1, y_2) = \left[\int_G \lambda(g) f(g, g^{-1}F(y)) \Delta(g)^{-1/2} \frac{g\psi_0}{\psi_0}(y_1) \psi(g^{-1}y_1, y_2) \right] \psi_0(y_1) \otimes \psi_0(y_2).$$

Again we see that the norm of $\pi_h(\omega)$ is equal to the norm of $\tilde{\pi}_h(\omega)$ where $\tilde{\pi}_h$ is a representation on $L^2(Y)$ given by:

$$(\tilde{\pi}_h(\omega)\Psi)(y) := \left[\int_G \lambda(g) f(g, g^{-1}F(y)) \Delta(g)^{-1/2} \psi(g^{-1}y) \frac{g\psi_0}{\psi_0}(y) \right] \psi_0(y).$$

Let ρ be a representation on $K(G, C_0(X))$ associated with a covariant representation (π, U) defined by morphism h . Then

$$\begin{aligned} (\rho(\hat{\omega})\Psi)(y) &= \int_G \lambda(g) (\pi(\hat{\omega}(g))U(g)\Psi)(y) = \int_G \lambda(g) \hat{\omega}(g)(F(y))(g\Psi)(y) = \\ &= \left[\int_G \lambda(g) \Delta(g)^{-1/2} f(g, g^{-1}F(y)) \frac{g\psi_0}{\psi_0}(y) \psi(g^{-1}y) \right] \psi_0(y). \end{aligned}$$

So $\|\rho(\hat{\omega})\| = \|\pi_h(\omega)\|$. And again $C^*(G \times X)$ is a kind of “smooth” crossed product, which is “smaller” than universal crossed product $C^*(G, C_0(X), \alpha)$.

Bisections as multipliers.

Let Γ be a differential groupoid, $\omega = \lambda \otimes \rho$ be a bidensity and B bisection. We define action of B on ω by:

$$(B\omega)(Bx)(Bv \otimes Bw) := \omega(x)(v \otimes w),$$

where $v \in \Lambda^{max} T_x^l \Gamma$, $w \in \Lambda^{max} T_x^r \Gamma$.

This is well defined since $BT_x^l \Gamma = T_{Bx}^l \Gamma$ and $BT_x^r \Gamma = T_{Bx}^r \Gamma$. Choose some ω_0 and for a bisection B define function $b : \Gamma \rightarrow R$ by $B\omega_0 =: b\omega_0$. Then b is nonvanishing and smooth. If $\omega = f\omega_0$ then $B\omega = B(f)\omega_0$ for $(Bf)(x) = f(B^{-1}x)b(x)$.

Since B acts as a diffeomorphism of Γ , it defines an unitary operator on $L^2(\Gamma)$ by: $(B\psi)(Bx)(Bv) := \psi(x)(v)$, $v \in \Lambda^{max} T_x \Gamma$ and ψ -smooth, half density on Γ . Choosing $\Psi_0 := \rho_0 \otimes \nu_0$ as in Prop. 4.13 we have $B\Psi_0 = b\Psi_0$ for the same function b .

Lemma 5.5 *Let B be a bisection and $\omega_1, \omega_2 \in \mathcal{A}(\Gamma)$, then $\omega_1^*(B\omega_2) = (s(B)\omega_1)^*\omega_2$.*

Proof: From $(B_1 B_2)\omega = B_1(B_2\omega)$ and $Bs(B) = \Gamma^0$ we have $s(b)(x) = \frac{1}{b(Bx)}$ where $s(b)$ is defined by $s(B)\omega_0 =: s(b)\omega_0$. Also from definition of ω_0 and b is easy to see that in fact b is defined by:

$$b(Bx)\rho_0(Bx)(Bw) = \rho_0(x)(w), \quad w \in \Lambda^{max} T_x^r \Gamma,$$

so $b(xy) = b(x)$. With the usual notation, the LHS is equal:

$$(\omega_1^*(B\omega_2))(x) = \left[\int_{F_l(x)} \lambda_0^2(y) f_1^*(y) (Bf_2)(s(y)x) \right] \omega_0(x). \text{ We compute:}$$

$$\int_{F_l(x)} \lambda_0^2(y) f_1^*(y) (Bf_2)(s(y)x) = \int_{F_l(x)} \lambda_0^2(y) \overline{f_1(s(y))} b(s(y)x) f_2(B^{-1}(s(y)x))$$

$$\text{And the RHS: } ((s(B)\omega_1)^*\omega_2)(x) = \left[\int_{F_r(x)} \rho_0^2(z) (s(B)f_1)^*(xs(z)) f_2(z) \right] \omega_0(x).$$

$$\begin{aligned} &\int_{F_r(x)} \rho_0^2(z) (s(B)f_1)^*(xs(z)) f_2(z) = \int_{F_r(x)} \rho_0^2(z) \overline{(s(B)f_1)(zs(x))} f_2(z) = \\ &= \int_{F_r(a)} \rho_0^2(z) s(b)(zs(x)) \overline{f_1(s(B)^{-1}zs(x))} f_2(z) = \int_{F_r(a)} \rho_0^2(z') \frac{1}{b(Bz')} \overline{f_1(Bz')} f_2(z'x), \end{aligned}$$

where $a := e_L(x)$.

Now since B induces diffeomorphism $F_r(a) \rightarrow F_r(a)$ we can rewrite this integral as:

$$\int_{F_r(a)} \rho_0^2(z') b(z') \overline{f_1(z')} f_2(B^{-1}z'x) = \int_{F_l(x)} \lambda_0^2(y) b(s(y)) \overline{f_1(s(y))} f_2(B^{-1}s(y)x)$$

and since $b(s(y)) = b(s(y)x)$ this is equal to LHS.

■

Proposition 5.6 Let Γ be a differential groupoids and B a bissection of Γ . Then for any morphism $h : \Gamma \rightarrow \Gamma'$:

- a) $(\hat{h}(B\omega_1))\omega_2 = h(B)(\hat{h}(\omega_1)\omega_2)$, $\omega_1 \in \mathcal{A}(\Gamma)$, $\omega_2 \in \mathcal{A}(\Gamma')$
b) $\pi_h(B\omega_1) = h(B)\pi_h(\omega_1)$, $\omega_1 \in \mathcal{A}(\Gamma)$.

Proof: Choose ω_0, ω'_0 and let $\omega_1 = f_1\omega_0, \omega_2 = f_2\omega'_0$. Let B be a bissection. $B\omega_1 = (Bf_1)\omega_0, (Bf_1)(x) = f_1(B^{-1}x)b(x)$ and $h(B)\omega_2 = (h(B)f_2)\omega'_0, (h(B)f_2)(z) = f_2((h(B))^{-1}z)h(b)(z)$.

The left hand side of a):

$$(\hat{h}(B\omega_1)\omega_2)(z) = \left[\int_{F_l(c)} \lambda_0^2(x)(Bf)(x)t_h(x,y)f_2(y) \right] \omega'_0(z),$$

$$\int_{F_l(c)} \lambda_0^2(x)(Bf)(x)t_h(x,y)f_2(y) = \int_{F_l(c)} \lambda_0^2(x)b(x)f(B^{-1}x)t_h(x,y)f_2(y),$$

where $t_h(x,y) = (x,z)$.

And the right hand side: $(h(B)(\hat{h}(\omega_1)\omega_2))(z) = h(b)(z)(f_1 *_h f_2)((h(B))^{-1}z)\omega'_0(z)$,

$$h(b)(z)(f_1 *_h f_2)((h(B))^{-1}z) = h(b)(z) \int_{F_l(c')} \lambda_0^2(x')f_1(x')t_h(x',y')f_2(y'),$$

where $t_h(x',y') = (x', (h(B))^{-1}z)$.

Let $x_1 \in B$ be such that $e_L(x_1) = c$. Then $x' \mapsto x_1x' = Bx'$ is a diffeomorphism $F_l(c') \rightarrow F_l(c)$. Using this fact we can rewrite the last expression as: $h(b)(z) \int_{F_l(c)} \lambda_0^2(x)f_1(s(x_1)x)t_h(s(x_1)x,y')f_2(y')$, where $t_h(s(x_1)x,y') = (s(x_1)x, (h(B))^{-1}z)$. The situation is illustrated on the figure:

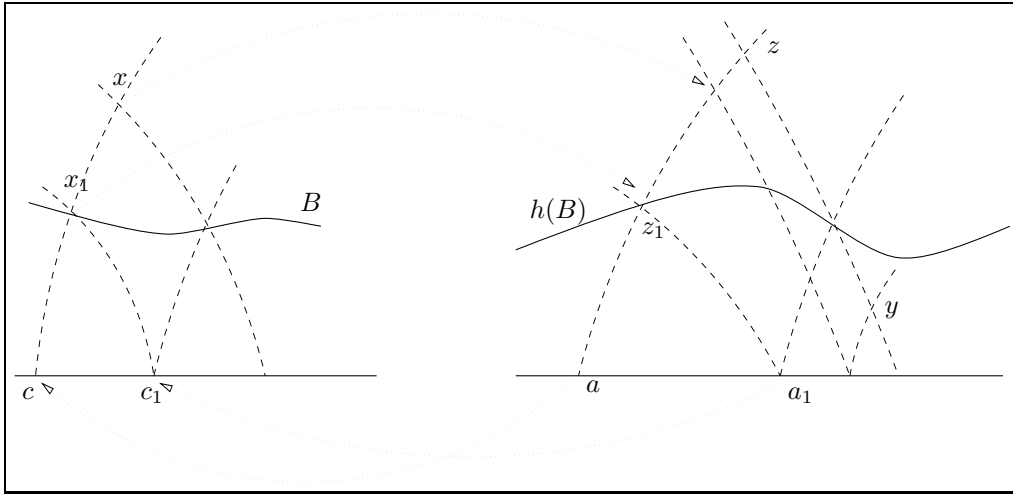


Figure 5.5:

Now we use the following:

- Lemma 5.7** a) $t_h(B \times id) = (B \times h(B))t_h$
b) $b(Bx)t_h(Bx,y) = t_h(x,y)h(b)(h(B)m_h(x,y))$ for any $(x,y) \in \Gamma \times_h \Gamma'$.

From the point a) we have $y' = y$ and from the point b): $b(x)t_h(x,y) = h(b)(z)t_h(B^{-1}x,y)$. The statement a) is proven.

b) If we write $\Psi = \psi\Psi_0$ then the integrals appearing in the formula b) are the same as in a), so b) is proven.

Proof of the lemma: a) Let $(x,y) \in \Gamma \times_h \Gamma'$, $a := e'_L(y)$. $Bx = x_1x$ for a unique point $x_1 \in B$.

$$t_h(B \times id)(x,y) = t_h(x_1x,y) = (x_1x, h_a^R(x_1x)y) = (x_1x, h_b^R(x_1)h_a^R(x)y),$$

for $b := e'_L(h_a^R(x))$. But this is equal to

$$(x_1x, h(B)h_a^R(x)y) = (B \times h(B))(x, h_a^R(x)y) = (B \times h(B))t_h(x,y).$$

b) Let $(x, y) \in \Gamma \times_h \Gamma'$, $c := e'_R(y)$. $\rho_0(x) \otimes \rho'_0(y)$ defines half density on $T_{(x,y)}(\Gamma \times_h \Gamma'_c)$.

$$\mathbf{t}_h(B \times id)(\rho_0(x) \otimes \rho'_0(y)) = b(Bx)\mathbf{t}_h(\rho_0(Bx) \otimes \rho'_0(y)) = b(Bx)t_h(Bx, y)(\lambda_0(Bx) \otimes \rho'_0(z')),$$

where $\mathbf{t}_h(Bx, y) = (Bx, z')$, i.e. $z' = h(B)m_h(x, y)$.

From the other side:

$$(B \times h(B))\mathbf{t}_h(\rho_0(x) \otimes \rho'_0(y)) = (B \times h(B))t_h(x, y)(\lambda_0(x) \otimes \rho'_0(z)) = t_h(x, y)h(b)(z')(\lambda_0(Bx) \otimes \rho'_0(z')).$$

■

Corollary 5.8 $B \in MC^*(\Gamma)$ and $C^*(h)B = h(B)$. Indeed, from statement b) of the previous proposition: $\|\pi_h(B\omega)\| \leq \|\pi_h(\omega)\|$ so also $\|B\omega\|_{C^*} \leq \|\omega\|_{C^*}$, so B can be extended to a bounded linear mapping of $C^*(\Gamma)$. Moreover from Lem. 5.5 $B \in MC^*(\Gamma)$ and $B^* = s(B)$. From the point a) of the previous proposition $C^*(h)(B) = h(B)$.

Functions on Γ^0 as affiliated elements.

Let g be a smooth function on Γ^0 . Define the action of g on bidensities by the formula $(g\omega)(x) := g(e_L(x))\omega(x)$. This is clearly a linear mapping. The following lemma is easy to prove:

Lemma 5.9 a) $\omega_1^*(g\omega_2) = (g^*\omega_1)^*\omega_2$, $\omega_1, \omega_2 \in \mathcal{A}(\Gamma)$ and g^* is a complex conjugation of g .

b) For $h : \Gamma \rightarrow \Gamma'$ - a morphism of differential groupoids let $h(g) : E' \rightarrow C$ be defined by: $h(g)(a) := g(f_h(a))$, then

$$\hat{h}(g\omega_1)\omega_2 = h(g)\hat{h}(\omega_1)\omega_2, \quad \omega_1 \in \mathcal{A}(\Gamma), \omega_2 \in \mathcal{A}(\Gamma')$$

$\pi_h(g\omega_1)\psi = h(g)\pi_h(\omega_1)\psi$, $\omega_1 \in \mathcal{A}(\Gamma), \psi \in L^2(\Gamma')$ - smooth with compact support and $h(g)$ is viewed as operator on $L^2(\Gamma')$ by multiplication.

If g is bounded then $\|h(g)\| \leq \sup|g|$.

■

From the lemma it follows that if g is smooth and bounded then g defines multiplier of $C^*(\Gamma)$. Moreover, using morphism $l : \Gamma \rightarrow \Gamma \times \Gamma$ one can see that $\|g\|_{C^*} = \sup|g|$, so we get isometric *-homomorphism from algebra of continuous, bounded function on Γ^0 to a multiplier algebra $MC^*(\Gamma)$.

Now we prove that continuous functions on Γ^0 are affiliated to $C^*(\Gamma)$. Start with the following:

Lemma 5.10 Let g be a continuous, bounded function on Γ^0 and assume that $f(a) \neq 0$ for $a \in \Gamma^0$. Then $gC^*(\Gamma)$ is dense in $C^*(\Gamma)$.

Proof: Take any $\omega \in \mathcal{A}(\Gamma)$ and let g_0 be a smooth function on Γ^0 with compact support such that $g_0|_{e_L(\text{supp } \omega)} = 1$. Then $\frac{g_0}{g}$ is bounded and continuous, so it defines a multiplier of $C^*(\Gamma)$. We have $g(\frac{g_0}{g}\omega) = (g\frac{g_0}{g})\omega = g_0\omega = \omega$. Since $\mathcal{A}(\Gamma)$ is dense in $C^*(\Gamma)$ this proves the assertion.

■

Now let g be a continuous function on Γ^0 . Define $z_g := \frac{g}{\sqrt{1+|g|^2}}$. Then z_g is a multiplier of $C^*(\Gamma)$, $\|z_g\| \leq 1$ and $(1 - z_g^*z_g)^{1/2}C^*(\Gamma)$ is dense in $C^*(\Gamma)$. So defining $T_g : C^*(\Gamma) \supset D(T_g) \rightarrow C^*(\Gamma)$ by:

$$(x \in D(T_g) \text{ and } y = T_g x) \iff (\exists a \in C^*(\Gamma) : x = (1 - z_g^*z_g)^{1/2}a \text{ and } y = z_g a)$$

we get an element affiliated with $C^*(\Gamma)$. (see appendix E). If g is smooth and $\omega \in \mathcal{A}(\Gamma)$ then $T_g\omega = g\omega$.

Final remarks.

The category of differential groupoids introduced by S. Zakrzewski led us to functorial construction of C^* -algebra of differential groupoid. It seems to be natural and generalize several well known examples. Will it be useful? It is almost certain that any double Lie group (so for example any Iwasawa decomposition) leads to a quantum group on the C^* algebra level. On the other hand, Iwasawa decompositions (or very similar to them) defines Poisson Lie structures on semidirect products, among them is Poincare group, so by the groupoid approach we can get corresponding quantum groups (if they exist) directly on the C^* level. However Poisson Lie structures coming from double Lie groups don't exhaust all possible Poisson structures and it is necessary to investigate more general situations (see Appendix A).

We end this work with the list of open questions:

1. In investigations of foliations we frequently meet groupoids which are not Hausdorff manifolds. Is it possible to generalize effectively our approach to such situations ?
2. Is there any connections between modular functions (or more precisely its cohomology class) and equality $C_{red}^*(\Gamma) = C^*(\Gamma)$?
3. Are sections of Lie algebroid of a differential groupoid affiliated with $C_{red}^*(\Gamma)$ or $C^*(\Gamma)$?
4. If we agree that morphisms of differential groupoids are relations, this leads us to a new notion of morphisms between Lie algebroids. Is this notion more natural or more useful in investigations of Lie algebroids ?

6 Appendixes.

Appendix A: Lie groupoids and C*-algebras

This is a part of a preliminary version of introduction left by S. Zakrzewski.

In this work we construct a functor from the category of smooth groupoids (with suitably defined morphisms) to the category of C*-algebras (with morphisms defined as in the context of *locally compact noncommutative spaces*, cf for instance [7]). The existence of such a functor was expected, once it was constructed for finite groupoids and since it has become clear what is the C*-algebra of a smooth groupoid.

We recall the definition of a smooth groupoid (called also *Lie groupoid* or *differential groupoid*) below. Apart from the standard definition, we recall our definition of groupoids as ‘algebras in the category of binary relations’, given previously in [1, 2], since it is crucial for the correct understanding of morphisms of groupoids in our sense.

In the rest of this introduction, let us explain the role of our construction in establishing relations between ‘classical’ and ‘quantum’ theories. Recall that symplectic manifolds correspond to (play a similar role as) Hilbert spaces (possibly projective) and symplectic diffeomorphisms correspond to unitaries.

In order to have a procedure which relates some concrete symplectic manifolds to some concrete Hilbert spaces and some concrete symplectic diffeomorphisms to unitaries one has to consider more special situation. Suppose we are given a manifold Q (playing the role of ‘configurations’). We have then immediately the corresponding phase space $S = T^*Q$ and also the Hilbert space $H = L^2(Q)$ (of square-integrable complex half-densities on Q). To any diffeomorphism ϕ of Q there corresponds a symplectomorphism $u := \phi_*$ (the push-forward of covectors) and also a unitary operator $U := \phi_*$ (the push-forward of half-densities). It is clear that in these circumstances we have a 1–1 correspondence between such u ’s and U ’s, illustrated by the following diagrams:

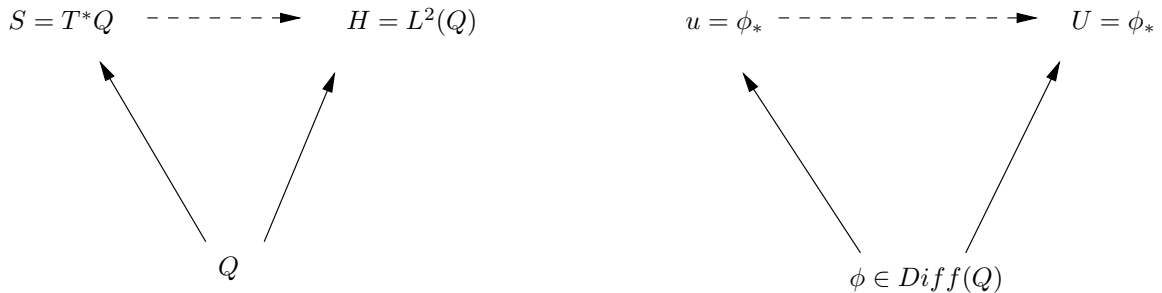


Figure 6.6:

We see that the transition from the classical level to the quantum level is possible in this case due to the common ‘configuration level’:

The symplectic diffeomorphisms of T^*Q being just the natural lifts of diffeomorphisms of Q are said to be *point transformations*. It turns out that not only these can be ‘quantized’. Namely, as a second step consider *phase shifts* of T^*Q , that is symplectic diffeomorphisms v of T^*Q of the form

$$T^*Q \ni \xi \mapsto \xi + df(\pi(\xi)) \in T^*Q,$$

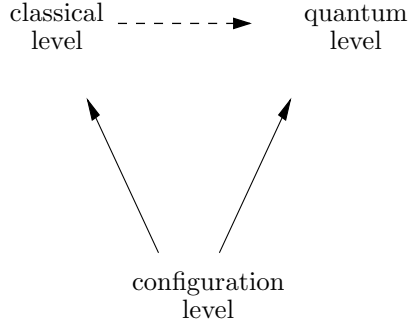


Figure 6.7:

where f is a smooth function on Q and $\pi: T^*Q \rightarrow Q$ is the cotangent bundle projection. It is natural to associate with f also the unitary operator V in $L^2(Q)$ of multiplication by e^{if} . Symbolically, we have

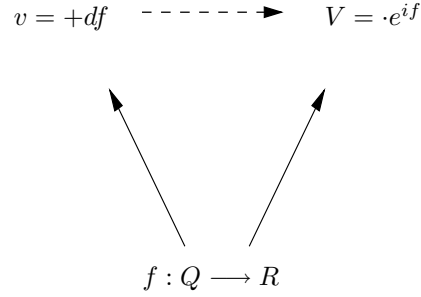


Figure 6.8:

It means that we can associate (projective) unitaries with phase shifts. Moreover, symplectic diffeomorphisms of the form vu form a group, which can be then naturally mapped into the unitary operators by the rule

$$vu \mapsto VU$$

(it works modulo the phase factor). What we here obtain is (essentially) the quantization of symplectic diffeomorphisms which preserve the natural polarization of the cotangent bundle (i.e. map fibers onto fibers). In fact, to construct (projective) H from S , the polarization is sufficient (the change of the lagrangian section playig the role of the ‘zero section’ is then implemented by the corresponding unitary transformations of type V).

We may summarize the above discussion as follows. A symplectic manifold S may serve to construct a quantum-mechanical Hilbert space H if S comes from configurations, $S = T^*Q$, or if at least S is equipped with projection on Q with lagrangian fibers (which essentially means that S is equipped with a polarization). Then a symplectic diffeomorphism of S may serve to construct a unitary operator in H if it is a point transformation, or, at least if it preserves the polarization.

Now the point is that such important for quantum mechanics structures as operator algebras (in particular, C^* -algebras) have also classical counterparts, namely symplectic groupoids. In this context, above diagrams have the following form:

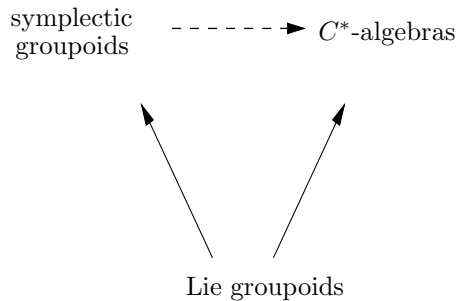


Figure 6.9:

whose concrete realization is

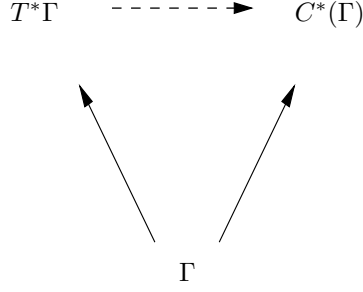


Figure 6.10:

Here Γ is a Lie groupoid, $T^*\Gamma$ is its cotangent (symplectic) groupoid and $C^*(\Gamma)$ is the C^* -algebra of Γ . Similarly, for morphisms, we shall have (as a result of the present paper)

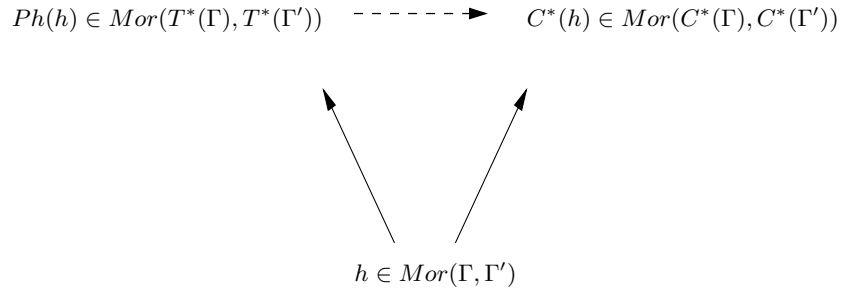


Figure 6.11:

This corresponds to the ‘point case’. There is also the second step, admitting also ‘phase shifts’. It consists in considering symplectic groupoids, which are projectable on Lie groupoids (in the sense that Γ is a cotangent bundle of some manifold Q and the multiplication relation projects onto a Lie groupoid multiplication relation on Q). It turns out that the symplectic groupoid structure of Γ is the cotangent lift of the Lie groupoid structure, shifted by a ‘2-cocycle’ and the previous diagram is generalized to

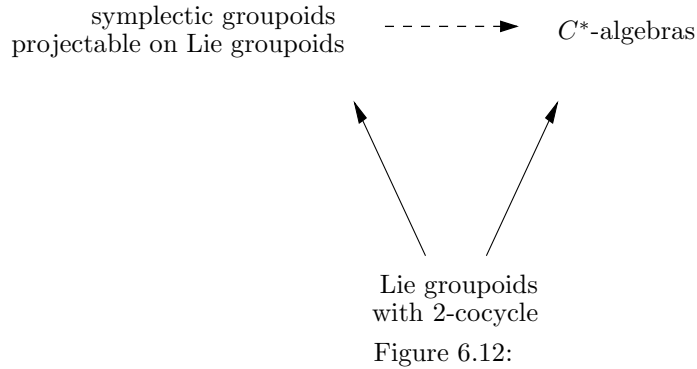


Figure 6.12:

The situation with 2-cocycles will be described in another paper.

Appendix B: Cocycles and one parameters group on $C_{red}^*(\Gamma)$.

Definition 6.1 A (smooth) one cocycle on Γ is a smooth function $\sigma : \Gamma \longrightarrow C$ which satisfies condition: $(x, y) \in \Gamma^2 \Rightarrow f(xy) = f(x)f(y)$.

For $x \in \Gamma^0$ we have $\sigma(x) = \sigma(xx) = \sigma(x)^2$ so $\sigma|_{\Gamma^0}$ is either 1 or 0. We assume that $\sigma|_{\Gamma^0} = 1$, then for any $x \in G$, $\sigma(x) \neq 0$. From now on the word *cocycle* will mean smooth, nonvanishing one cocycle.

Examples 6.2 a) If f is a non vanishing, smooth function on E then $\sigma(x) := \frac{f(e_L(x))}{f(e_R(x))}$ is a one cocycle on Γ .

b) Δ given in Example 4.14b) is a one cocycle on Γ .

c) Let $(G; A, B)$ be a double Lie group. The decomposition $G = AB$ defines decomposition $\mathfrak{g} := \mathfrak{a} \oplus \mathfrak{b}$ of a Lie algebra of G . Then operators of the adjoint representation of G can be written as $Ad_g := \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ for $\alpha_1 : \mathfrak{a} \rightarrow \mathfrak{b}$, $\alpha_2 : \mathfrak{b} \rightarrow \mathfrak{a}$, etc. Then the function $Q(g) := |\frac{\det Ad_g}{\det \alpha_1 \det \alpha_4}|$ is a one cocycle on G_A and G_B . This function plays an important role in a definition of the quantum group structure on $C^*(G_A)$.

For a cocycle σ we define the mapping $\sigma : \mathcal{A} \ni \omega \mapsto \sigma(\omega) := \sigma\omega \in \mathcal{A}$. Note the following:

Lemma 6.3 1. σ is a homomorphism of \mathcal{A}

2. If $|\sigma| = 1$ then σ is a $*$ -homomorphism of \mathcal{A} .

Proof: Using the formulas given in Col. 4.5 one performs simple computations.

■

In this way for any cocycle $\sigma : \Gamma \rightarrow]0, \infty[$ the mapping $R \ni t \mapsto \sigma^{it}$ defines one parameter group of $*$ -automorphisms of \mathcal{A} . Also it defines strongly continuous one parameter group $U_\sigma(t)$ of unitary operators on $L^2(\Gamma) : (U_\sigma(t)\psi)(x) := \sigma^{it}(x)\psi(x)$. Moreover $\pi_{id}(\sigma_t\omega) = U_\sigma(t)\pi_{id}(\omega)U_\sigma(-t)$ for $\omega \in \mathcal{A}(\Gamma)$.

Proposition 6.4 For any smooth cocycle $\sigma : \Gamma \rightarrow]0, \infty[$, σ_t is strongly continuous one parameter group on $C_{red}^*(\Gamma)$. (see appendix E)

Proof: First note that if $U(t)$ is a one parameter group of unitaries on a Hilbert space H then the set $C := \{a \in B(H) : \lim_{t \rightarrow 0} \|U(t)aU(-t) - a\| = 0\}$ is closed $*$ -subalgebra of $B(H)$. We will show that $\lim_{t \rightarrow 0} \|U_\sigma(t)\pi_{id}(\omega)U_\sigma(-t) - \pi_{id}(\omega)\| = 0$ for $\omega \in \mathcal{A}(\Gamma)$.

Now take some ω_0 and let $\|\cdot\|_l, \|\cdot\|_r, \|\cdot\|_0$ denote the associated norms on $\mathcal{A}(\Gamma)$. From prop. 4.13 we have $\|\pi_{id}(\omega)\| \leq \|\omega\|_0$ and

$$\|U_\sigma(t)\pi_{id}(\omega)U_\sigma(-t) - \pi_{id}(\omega)\| = \|\pi_{id}(\sigma_t(\omega) - \omega)\| \leq \|\sigma_t(\omega) - \omega\|_0.$$

$\|\sigma_t(\omega) - \omega\|_l = \sup_{a \in \Gamma^0} \int_{F_l(a)} \lambda_0(x)^2 |\sigma(x)^{it}f(x) - f(x)|$ where $\omega = f\omega_0$. Since $\text{supp } \omega$ is compact, one can find M such that $|\log \sigma(x)| \leq M$ for $x \in \text{supp } \omega$. So for $x \in \text{supp } \omega : |\sigma(x)^{it}f(x) - f(x)| \leq |t|M|f(x)|$ and $\|\sigma_t(\omega) - \omega\|_l \leq |t|M\|\omega\|_l$. In the same way we have $\|\sigma_t(\omega) - \omega\|_r \leq |t|M\|\omega\|_r$ and as a consequence: $\lim_{t \rightarrow 0} \|U_\sigma(t)\pi_{id}(\omega)U_\sigma(-t) - \pi_{id}(\omega)\| = 0$ for $\omega \in \mathcal{A}(\Gamma)$. Now the assertion follows from the remark on the beginning of the proof.

■

Let σ_i be an analytic generator of σ_t (see appendix E).

Proposition 6.5 $\mathcal{A}(\Gamma)$ is a core for σ_i and for $\omega \in \mathcal{A}(\Gamma)$ $\sigma_i(\omega) = \sigma^{-1}\omega$.

Proof: Start with the following:

Lemma 6.6 Let $\sigma : \Gamma \rightarrow]0, \infty[$ be a smooth cocycle and $\omega \in \mathcal{A}(\Gamma)$. The function $C \ni z \mapsto \sigma^{iz}\omega \in C^*(\Gamma)$ (or $C_{red}^*(\Gamma)$) is an entire analytic.

Proof: Choose some ω_0 , then straightforward computation shows that $\lim_{h \rightarrow 0} \|\frac{\sigma^{i(z+h)}\omega - \sigma^{iz}\omega}{h} - i \log(\sigma)\sigma^{iz}\omega\|_0 = 0$. Since $\|\omega\|_{C_{red}^*} \leq \|\omega\|_{C^*} \leq \|\omega\|_0$ we have the desired result.

■

From the lemma we have that $\mathcal{A}(\Gamma) \subset D(\sigma_i)$ and $\sigma_i(\omega) = \sigma^{-1}\omega$. For $a \in B(L^2(\Gamma))$ we set:

$$R_n(a) := \frac{n}{\sqrt{\pi}} \int_R dt e^{-n^2 t^2} U_\sigma(t) a U_\sigma(-t)$$

(since $U_\sigma(t)$ is strongly continuous $U_\sigma(t)aU_\sigma(-t)$ is σ -weakly continuous, and the integral is defined by the property that for any normal functional φ on $B(L^2(\Gamma))$: $\varphi(R_n(a)) = \frac{n}{\sqrt{\pi}} \int_R dt e^{-n^2 t^2} \varphi(U_\sigma(t)aU_\sigma(-t))$). We know [7] that the set $\{R_n(a) : n \in \mathbb{N}, a = \pi_l(\omega), \omega \in \mathcal{A}(\Gamma)\}$ is a core for σ_i . So it is enough to show that $R_n(\pi_l(\mathcal{A})) \subset \pi_l(\mathcal{A})$. Let $\sigma_n(x) := \frac{n}{\sqrt{\pi}} \int_R dt e^{-n^2 t^2} \sigma(x)^{it}$. It is clear that $\sigma_n \mathcal{A} \subset \mathcal{A}$. We claim that $R_n(\pi_l(\omega)) = \pi_l(\sigma_n \omega)$. Indeed, take any ψ -smooth, half density with compact support on Γ , then the equality $(\psi, R_n(\pi_l(\omega))\psi) = (\psi, \pi_l(\sigma_n \omega)\psi)$ follows from Fubini theorem.

■

Appendix C: Weights on $C_{red}^*(\Gamma)$

Choose some ρ_0 and real, smooth half density ν on Γ^0 . Such data define linear functional φ on \mathcal{A} as follows: write $\omega = f\omega_0$ and put $\varphi(\omega) := \int_{\Gamma^0} \nu^2 f$. Also they define linear mapping $\hat{\varphi} : \mathcal{A}(\Gamma) \ni \omega \mapsto f\rho_0 \otimes \nu \in L^2(\Gamma)$. Note the following:

Lemma 6.7 a) $\varphi(\omega^*\omega) = (\hat{\varphi}(\omega), \hat{\varphi}(\omega))$. (so φ is a positive, linear functional on $\mathcal{A}(\Gamma)$)
b) If π_{id} is the left regular representation of $\mathcal{A}(\Gamma)$ on $L^2(\Gamma)$ then: $\hat{\varphi}(\omega_1\omega_2) = \pi_{id}(\omega_1)\hat{\varphi}(\omega_2)$

Proof: Straightforward computation.

■

Now we assume that ν is non vanishing. Due to the above proposition we can identify the Hilbert space given by the GNS construction for φ and associated representation of \mathcal{A} with $L^2(\Gamma)$ and left regular representation. Next we show that φ can be extended to a weight on $C^*(\Gamma)$. We recall the following theorem [6]:

Theorem 6.8 Let π be a nondegenerate representation of a separable C^* algebra A on a separable Hilbert space H . Moreover let $\hat{\varphi} : A \supset D(\hat{\varphi}) \rightarrow H$ be a closed, densely defined linear mapping with the dense range such that $D(\hat{\varphi})$ is a left ideal of A and $\hat{\varphi}(ab) = \pi(a)\hat{\varphi}(b)$, $a \in A$, $b \in D(\hat{\varphi})$. Then the formula $\varphi(a^*a) := \begin{cases} (\hat{\varphi}(a), \hat{\varphi}(a)) & \text{if } a \in D(\hat{\varphi}) \\ \infty & \text{otherwise} \end{cases}$ defines a locally finite, lower semicontinuous weight on A .

■

First we are going to prove the following:

Proposition 6.9 $\hat{\varphi} : C_{red}^*(\Gamma) \rightarrow L^2(\Gamma)$ is closable.

Proof: We have to show, that if $\lim_{n \rightarrow \infty} \pi_l(\omega_n) = 0$ - in $BL^2(\Gamma)$ and $\lim_{n \rightarrow \infty} \hat{\varphi}(\omega_n) = \psi$ - in $L^2(\Gamma)$ then $\psi = 0$.

The groupoid inverse s is a diffeomorphism of Γ so it defines the unitary operator $S : L^2(\Gamma) \rightarrow L^2(\Gamma)$, also it defines linear, antimultiplicative bijection, which we denote also by $s : \mathcal{A} \rightarrow \mathcal{A}$. Let Δ be a modular function for (ρ_0, ν) , see Ex. 4.14 b) i.e. Δ is given by: $\Delta(x)(\rho_0 \otimes \nu)(x) = (\lambda_0 \otimes \nu)(x)$.

Lemma 6.10 $S\hat{\varphi}(\omega) = \hat{\varphi}\Delta s(\omega)$ $\omega \in \mathcal{A}$.

Proof: First let us note that $S(\rho_0 \otimes \nu) = \lambda_0 \otimes \nu$. Indeed, let $\{X^1, \dots, X^m\}$ be a basis in $T_x\Gamma$ such that $\{X_1, \dots, X^k\}$ is a basis in $T_x^l\Gamma$ (then of course $\{s(X^1), \dots, s(X^k)\}$ is a basis in $T_{s(x)}^r\Gamma$). Compute:

$$\begin{aligned} (S(\rho_0 \otimes \nu))(x)(X^1 \wedge \dots \wedge X^m) &:= (\rho_0 \otimes \nu)(s(x))(s(X^1) \wedge \dots \wedge s(X^m)) = \\ &= \rho_0(s(x))(s(X^1) \wedge \dots \wedge s(X^k)) \nu(e_R(s(x)))(e_R s(X^{k+1}) \wedge \dots \wedge e_R s(X^m)) = \\ &= \lambda_0(x)(X^1 \wedge \dots \wedge X^k) \nu(e_L(x))(e_L X^{k+1} \wedge \dots \wedge e_L X^m) = (\lambda_0 \otimes \nu)(x)(X^1 \wedge \dots \wedge X^m), \end{aligned}$$

since $\rho_0 s = \lambda_0$ and $e_R s = e_L$.

Now let $\omega_0 \omega = f\omega_0$. Then $(S\hat{\varphi}(\omega))(x) = (S(f\rho_0 \otimes \nu))(x) = f(s(x))(\lambda_0 \otimes \nu)(x)$ and $\Delta s(f\omega_0) = \tilde{f}\omega_0$ for $\tilde{f}(x) := \Delta(x)f(s(x))$ so

$$(\hat{\varphi}\Delta s(\omega))(x) = (\hat{\varphi}(\tilde{f}\omega_0))(x) = \tilde{f}(x)(\rho_0 \otimes \nu)(x) = \Delta(x)f(s(x))(\rho_0 \otimes \nu)(x) = f(s(x))(\lambda_0 \otimes \nu)(x).$$

■

For any $\omega \in \mathcal{A}$:

$$\begin{aligned} S\pi_{id}(\omega)S\psi &= S\pi_{id}(\omega)S(\lim_{n \rightarrow \infty} \hat{\varphi}(\omega_n)) = S(\lim_{n \rightarrow \infty} \pi_{id}(\omega)S\hat{\varphi}(\omega_n)) = \\ &= S(\lim_{n \rightarrow \infty} \pi_{id}(\omega)\hat{\varphi}(\Delta s(\omega_n))) = S(\lim_{n \rightarrow \infty} \hat{\varphi}(\omega\Delta s(\omega_n))) = \\ &= S(\lim_{n \rightarrow \infty} \hat{\varphi}(\Delta s(\omega_n s\Delta^{-1}(\omega)))) = S(\lim_{n \rightarrow \infty} S\hat{\varphi}(\omega_n s\Delta^{-1}(\omega))) = \lim_{n \rightarrow \infty} \pi_{id}(\omega_n)\hat{\varphi}(s\Delta^{-1}(\omega)) = 0. \end{aligned}$$

So $\pi_{id}(\omega)S\psi = 0$, for any $\omega \in \mathcal{A}(\Gamma)$. Since left regular representation is nondegenerate $S(\psi) = 0$ and $\psi = 0$.

■

We denote the closure of $\hat{\varphi}$ by the same symbol. So $\hat{\varphi} : C_{red}^*(\Gamma) \supset D(\hat{\varphi}) \rightarrow L^2(\Gamma)$ where $D(\hat{\varphi}) := \{a \in C_{red}^*(\Gamma) : a = \lim a_n, a_n \in \mathcal{A}(\Gamma), \text{ and there exists } x \in L^2(\Gamma), x = \lim \hat{\varphi}(a_n)\}$. It is also clear that the range of $\hat{\varphi}$ is dense in $L^2(\Gamma)$.

Lemma 6.11 *$D(\hat{\varphi})$ is a left ideal in $C_{red}^*(\Gamma)$ and $\hat{\varphi}(ab) = \pi_{id}(a)\hat{\varphi}(b)$ for $a \in C_{red}^*(\Gamma), b \in D(\hat{\varphi})$.*

Proof: Let $a \in C_{red}^*(\Gamma), b \in D(\hat{\varphi}), b = \lim b_n$ and $a = \lim a_m$ for some $a_m, b_n \in \mathcal{A}(\Gamma)$. Then $ab = \lim a_m b_n$, $\pi_{id}(a) = \lim \pi_{id}(a_m)$ and $x := \lim \hat{\varphi}(b_n)$. $\hat{\varphi}(a_m b_n) = \pi_{id}(a_m)\hat{\varphi}(b_n)$ and this sequence tends to $\pi_{id}(a)x$. It follows that $ab \in D(\hat{\varphi})$ and $\hat{\varphi}(ab) = \pi_{id}(a)\hat{\varphi}(b)$.

■

Using theorem 6.8 φ extends to a locally finite, lower semicontinuous weight on $C_{red}^*(\Gamma)$.

Now we prove that φ is a KMS-weight with $\sigma_t := |\Delta|^{-2it}$ as a modular group. Since Δ is non vanishing $|\Delta|$ is a smooth cocycle and σ_t is strongly continuous one parameter group on $C_{red}^*(\Gamma)$. Using the same arguments as in Prop. 6.5 one can show that $\mathcal{A}(\Gamma)$ is a core for $\sigma_{\frac{i}{2}}$ and for $\omega \in \mathcal{A}(\Gamma)$: $\sigma_{\frac{i}{2}}(\omega) = \frac{1}{|\Delta|}\omega$. Consider the following:

Lemma 6.12 *For $a \in D(\sigma_{\frac{i}{2}})$: $a \in D(\hat{\varphi}) \iff (\sigma_{\frac{i}{2}}(a))^* \in D(\hat{\varphi})$.*

Proof: For $\omega = f\omega_0 \in \mathcal{A}(\Gamma)$ we have: $(\sigma_{\frac{i}{2}}(\omega))^* = |\Delta|f^*\omega_0$ and $\hat{\varphi}((\sigma_{\frac{i}{2}}(\omega))^*) = |\Delta|f^*\rho_0 \otimes \nu$.

$$\begin{aligned} ||\Delta|f^*\rho_0 \otimes \nu||^2 &= \int_{\Gamma} (\rho_0^2(x) \otimes \nu^2(x)) |\Delta(x)|^2 |f(s(x))|^2 = \int_{\Gamma} (\lambda_0^2(x) \otimes \nu^2(x)) |\Delta(s(x))|^2 |f(x)|^2 = \\ &= \int_{\Gamma} (\rho_0^2(x) \otimes \nu^2(x)) |f(x)|^2 = ||\hat{\varphi}(\omega)||^2, \end{aligned}$$

where in the second equality we use S and in the third the definition of Δ . So for $\omega \in \mathcal{A}(\Gamma)$ we have:

$$||\hat{\varphi}((\sigma_{\frac{i}{2}}(\omega))^*)|| = ||\hat{\varphi}(\omega)||.$$

Now let $a \in D(\sigma_{\frac{i}{2}}), a = \lim \omega_n, \omega_n = f_n \omega_0 \in \mathcal{A}(\Gamma)$ and $\sigma_{\frac{i}{2}}(a) = \lim \sigma_{\frac{i}{2}}(\omega_n)$. So $a \in D(\hat{\varphi})$ is equivalent to a convergence of a sequence $\hat{\varphi}(\omega_n)$ and due to the above equality this is equivalent to a convergence of a sequence $\hat{\varphi}((\sigma_{\frac{i}{2}}(\omega_n))^*)$.

■

From the above lemma we have: $\varphi(a^*a) = (\hat{\varphi}(a), \hat{\varphi}(a)) = (\hat{\varphi}((\sigma_{\frac{i}{2}}(a))^*), \hat{\varphi}((\sigma_{\frac{i}{2}}(a))^*)) = \varphi(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*), a \in D(\sigma_{\frac{i}{2}})$. So to prove that φ is a KMS weight it remains to show that $\varphi\sigma_t = \varphi$ and this is straightforward computation. In this way we prove:

Proposition 6.13 *Let ρ_0 be smooth, real, non vanishing, rightinvariant half density on Γ and ν be real, smooth, non vanishing half density on Γ^0 . Let Δ be a modular function for this pair. Then the formula: $\varphi(f\omega_0) := \int_{\Gamma^0} f\nu^2$ defines locally finite, lower semicontinuous weight on $C_{red}^*(\Gamma)$ which is KMS weight with a modular group: $\sigma_t := |\Delta|^{-2it}$.*

■

Appendix D: Subgroupoids and homogenous spaces.

In this appendix we discuss the notion of subgroupoid and present some constructions related to it. Loosely speaking subgroupoid is subset closed with respect to multiplication and involution. We keep in mind the basic examples: subgroup of a group, subset of a set, equivalence relation and cartesian product $A \times A, A \subset X$ for pair groupoid $X \times X$. On the level of suitable algebras of functions in these examples we can relate to them the following constructions. Having a subgroup $H \subset G$ we can form the homogeneous space G/H together with an action of G . This action defines a morphism from G to a pair groupoid $G/H \times G/H$ which in turns defines a representation of group algebra of G . On the level of sets, if $A \subset X$ then restriction of a function on X to A defines a morphism from algebra of functions on X to an algebra of functions on A . For an equivalence relation $R \subset X \times X$ we can perform construction similar to the subgroup case and get a morphism from pair groupoid $X \times X$ to another pair groupoid. For the last example $A \times A \subset X \times X$ we get just subalgebra of functions on $X \times X$ which is loosely related to the whole algebra.

After these preliminary remarks we define objects and present constructions. According to the general line of this work we start from pure algebraic situation and later on add differential structure.

Definition 6.14 Let $\tilde{\Gamma} \subset \Gamma$. Define $\tilde{m} : \tilde{\Gamma} \times \tilde{\Gamma} \rightarrow \tilde{\Gamma}, \tilde{e} : \{1\} \rightarrow \tilde{\Gamma}$ and \tilde{s} by: $Gr(\tilde{m}) := Gr(m) \cap (\tilde{\Gamma} \times \tilde{\Gamma} \times \tilde{\Gamma}), Gr(\tilde{e}) := Gr(e) \cap (\tilde{\Gamma} \times \{1\})$ and $\tilde{s} := s|_{\tilde{\Gamma}}$. $\tilde{\Gamma}$ is a subgroupoid of Γ iff $(\tilde{\Gamma}, \tilde{m}, \tilde{s}, \tilde{e})$ is a groupoid.

We denote: $\tilde{E} := E \cap \tilde{\Gamma}$. For any subset $A \subset E$ the set $\tilde{\Gamma} := e_L^{-1}(A) \cap e_R^{-1}(A)$ is a subgroupoid of Γ . In the following we restrict our attention to subgroupoids of special kind which directly correspond to subgroups, subsets and equivalence relations. A subgroupoid $\tilde{\Gamma} \subset \Gamma$ is *horizontal* iff $\tilde{E} = E$; it is *vertical* iff for any $x \in \tilde{\Gamma}$ the fiber $F_l(x)$ is contained in $\tilde{\Gamma}$ (then of course also the right fiber $F_r(x)$ is contained in $\tilde{\Gamma}$.)

Lemma 6.15 *Let $\tilde{\Gamma} \subset \Gamma$ be a subgroupoid and let $i : \tilde{\Gamma} \rightarrow \Gamma$ be the inclusion map. Then:*

- a) $\tilde{\Gamma}$ is horizontal iff $i : \tilde{\Gamma} \rightarrow \Gamma$ is a morphism.
- b) $\tilde{\Gamma}$ is vertical iff $i^T : \Gamma \rightarrow \tilde{\Gamma}$ is a morphism.

Proof: The proof is straightforward application of the definitions. ■

Horizontal subgroupoids are also called *wide*.

Examples 6.16 a) If Γ is a group then every subgroup is horizontal subgroupoid and the only one vertical subgroupoid is Γ itself.

b) If Γ is a set then every subset is vertical subgroupoid and the only one horizontal subgroupoid is Γ itself.

c) If $\Gamma := X \times X$ is a pair groupoid, then every equivalence relation on X is a horizontal subgroupoid and the only one vertical subgroupoid is Γ itself. (since Γ has only one orbit.)

d) If Γ is a bundle of groups over E , then each subbundle is a horizontal subgroupoid and restriction of Γ to a subset of E is a vertical subgroupoid.

Let $\tilde{\Gamma}$ be a horizontal subgroupoid of Γ . Consider the relation R on Γ : $(x, y) \in R \iff s(x)y \in \tilde{\Gamma}$. It is easy to see that this is an equivalence relation. Let $\Gamma/\tilde{\Gamma} =: Y$ denote the set of equivalence classes.

Lemma 6.17 a) *The mapping $R \ni (x, y) \mapsto x \in \Gamma$ is surjective.*

b) *The mapping $f : Y \ni [x] \mapsto e_L(x) \in \Gamma^0$ is well defined and surjective.*

c) *Let $\Gamma \times_f Y := \{(x, y) \in \Gamma \times Y : e_R(x) = f(y)\}$. The mapping $g : \Gamma \times_f Y \ni (x, [x']) \mapsto [xx'] \in Y$ is well defined and surjective.*

d) *The relation $h : \Gamma \rightarrow Y \times Y$ given by: $Gr(h) := \{(g(x, y), y; x) : (x, y) \in \Gamma \times_f Y\}$ is a morphism from Γ to a pair groupoid $Y \times Y$.*

Proof: a) Let $x \in \Gamma$. Since $\tilde{\Gamma}$ is horizontal, $\tilde{E} = E$ and there exists $z \in \tilde{\Gamma}$ with $e_R(x) = e_L(z)$. Then $(x, xz) \in R$.

b) From the definition of R it is clear that if $(x, y) \in R$ then $e_L(x) = e_L(y)$, so f is well defined. Moreover for any $e \in E = \tilde{E}$ we have: $e = f([e])$.

c) It is clear that if $(x_1, x_2) \in R$ and $e_R(x) = f([x_1])$ then $(xx_1, x_2) \in R$ so the definition of g is correct. Moreover $g(f(y), y) = y$ for any $y \in Y$ so g is surjection.

d) Let us check, that h is a morphism of groupoids.

i) $hE = \text{diag}(Y \times Y)$.

If $e \in E$ and $f([x]) = e$ then $[ex] = [x]$, so $hE \subset \text{diag}(Y \times Y)$. Also for any $[x] \in Y$ we have: $([x], [x]; f([x])) \in h$.

ii) $hs = s'h$.

$(y_1, y_2, x) \in hs \iff (y_1, y_2; s(x)) \in h \iff e_L(x) = f(y_2) \text{ and } y_1 = s(x)y_2 \iff e_L(x) = f(y_2) \text{ and } e_R(x) = f(y_2) \text{ and } y_2 = xy_1 \iff (y_2, y_1; x) \in h \iff (y_1, y_2; x) \in s'h$.

iii) $hm = m'(h \times h)$.

$(y_1, y_2; x_1, x_2) \in hm \iff e_R(x_1) = e_L(x_2) \text{ and } (y_1, y_2; x_1x_2) \in h \iff e_R(x_1) = e_L(x_2) \text{ and } e_R(x_2) = f(y_2) \text{ and } y_1 = x_1x_2z_2 \Rightarrow e_R(x_1) = e_L(x_2) \text{ and } (x_2y_2, y_2; x_2) \in h \text{ and } (y_1, x_2y_2; x_1) \in h \Rightarrow (y_1, y_2; x_1, x_2) \in m'(h \times h)$.

Conversely, for $(y_1, y_2; x_1, x_2) \in m'(h \times h) \iff (y_1, y_3, x_1) \in h \text{ and } (y_3, y_2; x_2) \in h$ for some $y_3 \in Y$, so $e_R(x_1) = f(y_3)$, $e_R(x_2) = f(y_2)$ and $y_3 = x_2y_2$. Now $f(y_3) = e_L(x_2)$ so x_1, x_2 are composable and $(y_1, y_2; x_1x_2) \in h$.

■

In the differential setting we define:

Definition 6.18 Let Γ be a differential groupoid and $\tilde{\Gamma} \subset \Gamma$ be a submanifold. $\tilde{\Gamma}$ is a *differential subgroupoid* of Γ iff $(\tilde{\Gamma}, \tilde{m}, \tilde{s}, \tilde{e})$ is a differential groupoid.

Let $\tilde{\Gamma} \subset \Gamma$ be a horizontal subgroupoid of Γ and let the relation R , the mappings f and g be as above. We have the following:

Lemma 6.19 a) R is a submanifold of $\Gamma \times \Gamma$

b) The mapping: $R \ni (x, y) \mapsto x \in \Gamma$ is a surjective submersion.

c) If $\tilde{\Gamma}$ is closed then from a) and b) we have that Y is a manifold and then the mapping f is a surjective submersion.

d) $\Gamma \times_f Y$ is a submanifold of $\Gamma \times Y$ and g is a surjective submersion.

e) The relation h defined in the previous lemma is a morphism of differential groupoids.

Proof: a) Consider $\{(x, y) \in \Gamma^{(2)}; m(x, y) \in \tilde{\Gamma}\}$. Since m restricted to $\Gamma^{(2)}$ is surjective submersion and $\tilde{\Gamma}$ is a submanifold, this set is a submanifold in $\Gamma^{(2)}$ so also in $\Gamma \times \Gamma$. R is the image of this submanifold by the diffeomorphism: $(x, y) \mapsto (s(x), y)$.

b) It is clear that this mapping is smooth and surjective. Let $(x_0, y_0) \in R$ and $s(x_0)y_0 = z_0 \in \tilde{\Gamma}$. Let $x(t)$ be a curve through x_0 , then $e_R(x(t))$ is a curve in Γ^0 through $e_R(x_0) = e_L(z_0)$. Since \tilde{e}_L is a surjective submersion one can lift it to a curve $z(t) \in \tilde{\Gamma}$ through z_0 . Then $(x(t), x(t)z(t))$ is a curve in R through (x_0, y_0) .

c) This is clear, since $e_L = f\pi$ where $\pi : \Gamma \rightarrow Y$ is a surjective submersion.

d) Since $\Gamma \times_f Y = (e_R \times f)^{-1}(\text{diag}(\Gamma^0 \times \Gamma^0))$ and $e_R \times f$ is a surjective submersion it is clear that $\Gamma \times_f Y$ is a submanifold in $\Gamma \times Y$. On $\Gamma^{(2)}$ consider the relation $\tilde{R} := \text{id} \times R$ i.e. $(x_1, x_2; x_3, x_4) \in \tilde{R} \iff x_1 = x_3 \text{ and } (x_2, x_4) \in R$. Then this is a regular equivalence relation and $\Gamma^{(2)}/\tilde{R} = \Gamma \times_f Y$. Let $\tilde{p}_i : \Gamma^{(2)} \rightarrow \Gamma^{(2)}/\tilde{R}$ be the canonical projection. Then g is determined by: $\underline{m}\pi = g\tilde{p}_i$ where $\underline{m} : \Gamma^{(2)} \rightarrow \Gamma$ is a restriction of m to $\Gamma^{(2)}$. Since this restriction is surjective submersion g is a surjective submersion.

e) From the previous lemma we know that h is a morphism of groupoids. We have to check that $Gr(h)$ is a submanifold and two transversality condition $m' \uparrow (h \times h)$ and $h \uparrow e$ hold. Let $\tilde{Y} := \{(y, x) \in Y \times X : (x, y) \in \Gamma \times_f Y\}$. Then \tilde{Y} is a submanifold in $Y \times \Gamma$ and $\tilde{g} : \tilde{Y} \ni (y, x) \mapsto g(x, y) \in Y$ is a surjective submersion. So $Y \times \tilde{Y}$ is a submanifold in $Y \times Y \times \Gamma$ and $Gr(h) = (\pi_1 \times \tilde{g})^{-1}(\text{diag}(Y \times Y))$ where $\pi_1 : Y \times \tilde{Y} \ni (y, \tilde{y}) \mapsto y \in Y$. Since $\pi_1 \times \tilde{g}$ is a surjective submersion $Gr(h)$ is a submanifold in $Y \times \tilde{Y}$ and so in $Y \times Y \times \Gamma$.

Now we check transversality condition. First we take $m' \uparrow (h \times h)$. Since $m'(h \times h) = hm$ and m is a differentiable reduction we know [2] that $Gr(hm) = Gr(m'(h \times h))$ is a submanifold so it is enough to show that Tm' and $T(h \times h)$ and Pm' and $P(h \times h)$ have simple composition. Let $(v; u, w) \in Tm'T(h \times h)$ be vectors tangent to $Gr(m'(h \times h))$ at points $(y_1, y_2; x_1, x_2)$. We can write $v = (v_1, v_2)$ for $v_1 \in T_{y_1}Y$, $v_2 \in T_{y_2}Y$. Then we have $(v_1, v_3; u) \in Th$ and $(v_3, v_2; w) \in Th$ for some $v_3 \in T_{y_3}Y$ where y_3 is a unique point in Y such that $(y_1, y_3; x_1) \in h$ and $(y_3, y_2; x_2) \in h$. But then $v_3 = g(w, v_2)$ and since g is a mapping v_3 is uniquely determined. So Tm' and $T(h \times h)$ have a simple composition.

Now let $(\varphi_1, \varphi_2; \psi_1, \psi_2) \in Pm'P(h \times h)$ where $\varphi_1 \in T_{y_1}^*Y, \varphi_2 \in T_{y_2}^*Y, \psi_1 \in T_{x_1}^*\Gamma, \psi_2 \in T_{x_2}^*\Gamma$ and $(y_1, y_2; x_1, x_2) \in m'(h \times h)$. It is easy to see that in the case of pair groupoids

$$Gr(Pm) := \{(\varphi_1, \varphi_2; \varphi_1, \varphi_3, -\varphi_3, \varphi_2) : \varphi_1 \in T_{y_1}^*Y, \varphi_2 \in T_{y_2}^*Y, \varphi_3 \in T_{y_3}^*Y\}.$$

So our condition on $\varphi_1, \varphi_2, \psi_1, \psi_2$ is equivalent to existence of some $\varphi_3 \in T_{y_3}^*Y$ for unique y_3 - as above which satisfies $(\varphi_1, \varphi_3; \psi_1) \in Ph$ and $(-\varphi_3, \varphi_2; \psi_2) \in Ph$. From the second condition: $- < \varphi_3, g(u_2, w_2) > = - < \varphi_2, w_2 > + < \psi_2, u_2 >$ for any $(u_2, w_2) \in T_{x_2, y_2}(\Gamma \times_f Y)$. Since g is a submersion this equality determines φ_3 . In this way Pm' and $P(h \times h)$ have simple composition.

Now we check that $h \uparrow E$. The submanifold property is clear so we have to verify only simple compositions conditions. If $(v_1, v_2; w) \in T(he)$ then $v_1 = v_2$ and $(w, v_2) \in T_{(x, y)}(\Gamma \times_f Y)$ for some $w \in T_x\Gamma^0$. But it means that $w = e_R(w) = f(v_2)$ so w is uniquely determined. If $(\varphi_1, \varphi_2; \psi) \in P(he)$ then $\psi \in T_x^*E \subset T_x^*\Gamma$, $x = f(y)$, $< \varphi_1, g(w, v_2) > + < \varphi_2, v_2 > = < \psi, w >$ for any $w = f(v_2)$ and $v_2 \in T_{y_2}Y$. But $g(f(v_2), v_2) = v_2$ so the value of ψ on the vectors $f(v_2)$ is determined and since f is a submersion onto E this determines ψ uniquely. ■

Appendix E: C^* -algebras.

Multiplier algebra. Let A be a C^* -algebra. By $B(A)$ we denote the algebra of bounded linear mappings acting on A . Let $a, b \in B(A)$, we say that b is a *hermitian adjoint* of a and write $b = a^*$ if:

$y^*(ax) = (by)^*x$ for all $x, y \in A$. It follows that the set of those $a \in B(A)$ which have a hermitian conjugate is a C^* -algebra. This is a *multiplier algebra* of A and will be denoted by $M(A)$. A can be embedded into $M(A)$ via the left multiplication and the image is ideal in $M(A)$. $M(A) = A$ iff $1 \in A$.

C^ - category.* Let A, B be C^* algebras. *Morphism from A to B* is $*$ -homomorphism $\phi : A \rightarrow M(B)$ such

that the set $\phi(A)B$ is dense in B . That morphism can be composed follows from the fact that any such ϕ extends uniquely to a C^* -homomorphism from $M(A)$ to $M(B)$, this extension is defined by: $\hat{\phi}(m)(\phi(a)b) := \phi(ma)b$ for $m \in M(A)$, $a \in A$, $b \in B$. If $\phi_1 \in \text{Mor}(A, B)$, $\phi_2 \in \text{Mor}(B, C)$ then composition is defined by $\hat{\phi}_2\phi_1 : A \longrightarrow M(C)$. C^* algebras with above defined morphisms form a C^* category.

Affiliated elements. Let A be a C^* -algebra and $T : A \supset D(T) \longrightarrow A$ densely defined linear mapping. T is *affiliated with* A iff there exists $z \in M(A)$, $\|z\| \leq 1$ such that

$$(x \in D(T) \text{ and } y = Tx) \iff (\exists a \in A : x = (I - z^*z)^{1/2}a \text{ and } y = za).$$

If $1 \in A$ then the set of affiliated elements is equal to A .

Weights on C^ algebras.* Let A be a C^* algebra and A_+ be a set of positive elements. A weight φ on A is a mapping $\varphi : A_+ \longrightarrow [0, \infty]$ which satisfies: $\varphi(a + b) = \varphi(a) + \varphi(b)$, $\varphi(\lambda a) = \lambda\varphi(a)$ for any $\lambda \geq 0$ and $a, b \in A_+$. ($0 \cdot \infty = 0$). The weight φ is *densely defined* if the set $\{a \in A_+ : \varphi(a) < \infty\}$ is dense in A_+ . φ is *lower semi-continuous* if for any $t \in R_+$ the set $\{a \in A_+ : \varphi(a) \leq t\}$ is closed.

One parameter groups on C^ algebras.* The homomorphism $\sigma : R \longrightarrow \text{Aut}(A)$ - the group of $*$ -automorphisms of A such that $\|\sigma_t\| \leq 1$ for any $t \in R$ is called *one parameter group on A* . σ is *strongly continuous* iff for any $a \in A$ the function: $R \ni t \mapsto \sigma_t(a) \in A$ is continuous.

An analytic generator of a strongly continuous one parameter group σ_t on A is a linear mapping $\sigma_i : D(\sigma_i) \longrightarrow A$ defined in the following way: $a \in D(\sigma_i)$ iff there exists continuous function $f : \{z \in C : \text{Im}(z) \in [0, 1]\} \longrightarrow A$, analytic in the interior and such that $f(t) = \sigma_t(a)$ for all $t \in R$; then $\sigma_i(a) := f(i)$. It can be proved that analytic generators are densely defined, closed, multiplicative mappings.

KMS weights. Let σ_t be a strongly continuous, one parameter group on A , and φ densely defined, lower semi-continuous weight. φ is called *KMS-weight with modular group σ* iff $\varphi\sigma_t = \varphi$ and $\varphi(a^*a) = \varphi(\sigma_{\frac{i}{2}}(a)(\sigma_{\frac{i}{2}}(a))^*)$ for any $a \in D(\sigma_{\frac{i}{2}})$.

C^ dynamical systems and crossed products.* Let G be a locally compact group, A a C^* algebra and $\text{Aut}(A)$ -group of $*$ -automorphism of A . Let $G \ni g \mapsto \alpha_g \in \text{Aut}(A)$ be a strongly continuous group homomorphism. Then a *C^* dynamical system* is a triple (G, A, α) . A *covariant representation* of (G, A, α) is a pair (π, U) where π is nondegenerate $*$ -representation of A and U is strongly continuous, unitary representation of G such that $\pi(\alpha_g(x)) = U(g)\pi(x)U(g)^{-1}$. Let $K(G, A)$ denote the linear space of compactly supported continuous mapping from G to A . It follows that $K(G, A)$ is a normed $*$ -algebra if we define:

$$(f_1 f_2)(g) := \int_G \lambda(h) f_1(h) \alpha_h(f_2(h^{-1}g)), \quad (f*)(g) := \Delta(g)^{-1} (\alpha_g(f(g^{-1}))^*)$$

and $\|f\|_1 := \int_G \lambda(g) \|f(g)\|$, where λ is a left Haar measure and Δ corresponding modular function. Any covariant representation (π, U) defines a representation ρ of $K(G, A)$ by: $\rho(f) := \int_G \lambda(g) \pi(f(g)) U(g)$. One can show that $\|\rho(f)\| \leq \|f\|_1$. It follows that formula $\|f\| := \sup_{\rho} \|\rho(f)\|$ where the supremum is taken over all covariant representations defines C^* norm. The completion of $K(G, A)$ in this norm is *the crossed product $C^*(G, A, \alpha)$* .

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